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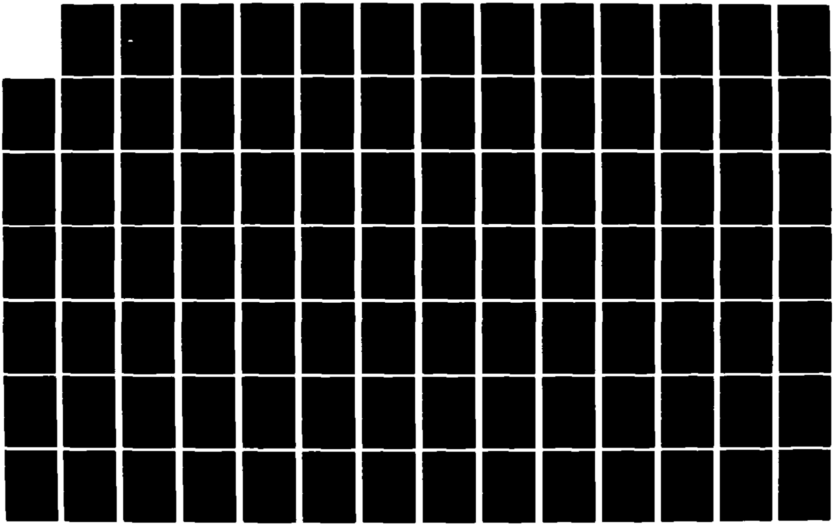

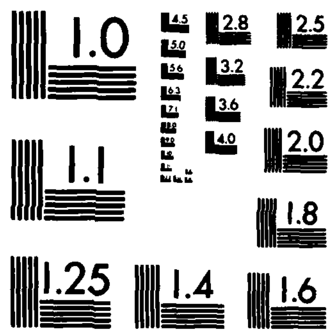

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A PROBABILISTIC MECHANICAL DESIGN AND ANALYSIS TECHNIQUE

Thomas L. Paez

September 1982

Final Report

Approved for public release; distribution unlimited.

AIR FORCE WEAPONS LABORATORY
Air Force Systems Command
Kirtland Air Force Base, NM 87117

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This final report was prepared by the Air Force Weapons Laboratory, Kirtland Air Force Base, New Mexico, under Job Order 57080501. Mr. Edsal Chappelle (NTSA) was the Laboratory Project Officer-in-Charge.

This technical report has been reviewed and is approved for publication.

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

This report summarizes practical procedures for the estimation of structural reliability. Starting from the elementary level, the theories of probability and random processes are reviewed. The concepts underlying reliability analysis are discussed. Reliability analyses of statically and dynamically loaded structures are discussed. Common assumptions used to make structural reliability analyses executable are given. And a criterion that can be used to determine which structural elements are critical to the reliability of a

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20. ABSTRACT (Continued)

structure is developed. The reliability analysis computer program, FSR, is discussed and the inputs required to run this program are summarized. The results of some numerical examples are summarized. Recommendations for future work in structural reliability analysis are provided.

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I. INTRODUCTION

In the design and analysis of engineering structures an accurate knowledge of inputs, structural material properties, and structural configuration is important, for it is only when this information is available that structural behavior can be accurately predicted and efficient structural designs can be produced. For practical reasons, it is necessary to make certain assumptions which simplify the analysis of structural systems. Only when simplifying assumptions are made is a practical solution possible for the mathematical equations governing the response of a structure. These generally reduce the accuracy of an analysis.

In the past, structural analyses and designs were performed using the simplifying assumption that nature is deterministic. Structural excitations, structural material properties and structural configurations were assumed to be nonrandom. As a result, some experimental and field experiences could not be easily explained. For example, it is difficult to account for the different failure strengths of different test specimens when nominally identical specimens are tested in nominally identical experiments. In some cases, it is only through consideration of random system behavior that real system experience can be quantitatively characterized.

In recent times, analytical techniques which take into account the random variation in mechanical systems and their inputs have been suggested. Numerous papers have been written on the subject of reliability of structures. Several of these deal with the problem of a static random load applied to a structure with random material properties. Others deal with the computation of response statistics of deterministic structures subjected to random dynamic loads.

The reliability of a structure is defined as the probability that the structure will perform in a satisfactory manner over a preestablished length of time. In computing structural reliability, we often define the reliability of a structure as the probability that it will survive, in a physical sense, over some intended life span. To discuss structural survival we must define the structural failure condition. Structural failure can be defined in a wide variety of ways. For example, at one extreme, catastrophic collapse of a structure can be used to define failure. At the other extreme, structural

1. PROBLEM DEFINITION

This report discusses methods for finding the probability of failure, or, equivalently, the reliability of a complex structure. In principle, it is possible to perform a reliability analysis on any structure which can be analyzed deterministically if the character of the random inputs and of the structure are known. In practice it may be difficult or expensive to execute some analyses.

In this report, failure is defined in terms of a peak value of response motion, or a peak value of a function of response motion, at one or more points on a structure. That is, failure is assumed to occur when some measure of the response exceeds a "failure level." Then the probability of failure is simply the chance that the measure of response surpasses the failure level. This response measure could be stress, strain, acceleration, moment, shear, etc., at a point. The purpose of this report is to demonstrate practical methods for computing this probability.

Note that it is important to maintain a consistent set of measures when considering the potential for failure at a point. For example, if a particular loading excites a peak stress of 30×10^7 Pa at a point on a structure, then it is necessary to determine the failure stress, in pascals, for the material at that point. This may be, for example, 35×10^7 Pa. If the response that a load excites is a moment measured in foot pounds, then the moment, in foot pounds, that can be carried at a point, before failure occurs, must be determined. If the response that an input excites is a force measured in kips, then the peak load, in kips, that a section can carry must be measured, etc. This report often associates the structural response excited by an input with the loading, and the load that can be carried at a point with the structural strength. So, using the first example in this paragraph, 30×10^7 Pa would be the load stress, and 35×10^7 Pa would be the structural strength.

This report shows how the analyst can consider any mode of failure in a structure as long as he can determine the load at any point on a stucture, and what level of load will excite a failure in the mode of interest. In this way, the analyst can use the techniques presented here to find the probability of failure in a nonlinear mode, but this will involve nonlinear structural analysis.

Considered in this report are: (1) static and dynamic loads on a structure; (2) the general case of many repetitions of a static load; and (3) steady state and transient dynamic loads.

II. FUNDAMENTAL CONCEPTS IN PROBABILITY, RANDOM PROCESSES AND RELIABILITY

This section provides the fundamental concepts necessary for the performance of a structural reliability analysis. First, the probability concepts used in computing the reliability of a structure at a point are outlined, and examples are given for the important normal distribution case. Second, the formulas necessary for obtaining reliability analysis inputs from random process parameters are presented. Finally, the guidelines for performing a reliability analysis of a complex structure are presented.

This section uses two concepts important in structural reliability analysis: structural load and structural strength. When a structure is excited by external forces, it executes a mechanical response. The overall response deformation of a structure induces loads in the structural members. These loads can be measured in units of stress, strain, moment, force, etc. When a loaded member possesses sufficient strength to support an applied force, then the member survives. For consistency, the strength of a member is measured in the same units used to measure the load. Even though load and strength can be measured in several different ways, this report usually refers to the load stress at a point and to the structural material strength. Our understanding is that load and strength can be measured in any appropriate way in an actual problem.

Note that, in the solution of a particular problem, the forces induced at specific points must be determined. Therefore, structural analyses must be performed. For most complicated structures, finite element analyses can be used to accurately determine the structural response at a point.

1. PROBABILITY CONCEPTS

a. Random experiment, random variable, PDF, CDF--When a mechanical experiment is performed, one or more outcomes can be measured and recorded. If an attempt is made to duplicate the experiment, the outcomes will vary from the first experiment to the next. The differences in the outcomes may be great or small, but they will always be present because of the randomness in nature. When randomness has a relatively small effect on experimental outcome, it can be ignored; otherwise it must be considered in design and analysis.

To discuss probability and randomness we define the random experiment. A random experiment is an experiment whose outcome is uncertain. For example, a random experiment might consist of flying an airplane in a specified way, under specific conditions, for a specified time duration. Some typical outcomes of this random experiment may be the following. The peak stress at one location near the root of the left wing experienced during the flight is equal to 3×10^8 Pa. Or, the peak acceleration at the center of gravity of the airplane experienced during the flight is equal to 5 g. Practically an infinite number of outcomes can be defined for a random experiment such as that given above. Each outcome constitutes a random event, and every well-defined collection of outcomes constitutes a random event. For example, a random event might be defined as follows. The peak acceleration that the center of gravity of the airplane experiences during the flight is in the range (4 g, 6 g]. This includes the uncountable infinity of outcomes in the range (4 g, 6 g].

The probability of an event is a number between zero and one, and it reflects the relative chance of occurrence of an event. An event which has no chance of occurrence has zero probability, and a sure event has probability one. For example, we might find as the result of analysis that the probability of the event defined above (i.e., that the peak acceleration the center of gravity of an airplane experiences during a flight is in the range (4 g, 6 g]) is equal to 0.85. This result means that "on the average," if we performed a large number of the random flight experiments, approximately 85 percent of these would yield a peak acceleration in the range (4 g, 6g]. We say "on the average," because the experimental outcomes are random and there is no guarantee the event in question will be realized 85 percent of the time whenever a finite sequence of experiments is performed. Further, note that the probability value of 0.85 is a quantity which relates to the structure of an experiment, and has nothing to do with the number of experiments performed. The probability of occurrence of two events which cannot occur simultaneously is the sum of the probabilities of the individual events.

The random variable is the fundamental entity used to quantify the outcomes of a random experiment. Generally, a capital letter denotes a random variable. So, for example, let X be the peak acceleration at the center of gravity of an airplane during the flight, with specified parameters, described previously. The random variable, X, can take certain values and we denote these using the lower case letters corresponding to the random variable. The values, x, that the random variable, X, can take are called the realizations of the random variable. The event defined previously (i.e., that the peak acceleration the center of gravity of an airplane experiences during a flight is in the range (4 g, 6 g]) is stated as follows in terms of the random variable.

$$4\ g < X \leq 6\ g \tag{1}$$

The probability of this event is expressed

$$P(4 < X \leq 6) \tag{2}$$

The probabilistic behavior of a random variable, X, can be defined using either of two functions. We start by defining the probability density function (pdf). The pdf of a random variable, X, is denoted $p_X(x)$, and this is defined implicitly using the following integral.

$$P(x_1 < X \leq x_2) = \int_{x_1}^{x_2} p_X(x)dx \tag{3}$$

The probability that a random variable, X, has a realization in $(x_1, x_2]$, as the result of a random experiment, is given by the above integral. Since a probability must be equal to or greater than zero, and a probability must be less than one, a valid pdf must satisfy the following equation.

$$p_X(x) \geq 0 \qquad -\infty < x < \infty \tag{4a}$$

$$\int_{-\infty}^{\infty} p_X(x)\ dx = 1 \tag{4b}$$

As the result of single random experiment, the realization of a random variable, X, cannot be within both of the intervals, $(x_1, x_2]$ and $(x_3, x_4]$, if the intervals do not overlap. Therefore, the probability that the realization lies in $(x_1, x_2]$ or $(x_3, x_4]$ is the probability that the realization is in $(x_1, x_2]$ plus the probability that the realization is in $(x_3, x_4]$. Using the symbol $\cup$ to denote the operation, "or," we can write

$$P(x_1 < X \leq x_2 \cup x_3 < X \leq x_4) = P(x_1 < X \leq x_2) + P(x_3 < X \leq x_4)$$

$$= \int_{x_1}^{x_2} p_X(x)dx + \int_{x_3}^{x_4} p_X(x)dx \qquad (5)$$

An alternate, and equivalent, means for defining the probabilistic behavior of a random variable is through use of the cumulative distribution function (cdf). The cdf of the random variable, X, is denoted $P_X(x)$, and is defined

$$P_X(x) = P(X \leq x) = \int_{-\infty}^{x} p_X(\alpha)d\alpha \qquad -\infty < x < \infty \qquad (6)$$

This is simply the chance that the realization of a random variable resulting from a random experiment is equal to or lower than x. Because of the restrictions on a probability and the requirements imposed on a pdf, we can write the following restrictions on a cdf

$$P_X(x) \geq 0 \qquad -\infty < x < \infty \qquad (7a)$$

$$P_X(-\infty) = 0, \ P_X(\infty) = 1 \qquad (7b)$$

We can also express the probability that the realization of X falls in the interval $(x_1, x_2]$ using the cdf. It is

$$P(x_1 < X \leq x_2) = P_X(x_2) - P_X(x_1) \qquad (8)$$

From this expression it is clear that the inequalities in the left hand expression and in Equation 3 take the form they do because of the way the cdf is defined in Equation 6. By differentiating the cdf in Equation 6 we obtain

$$p_X(x) = \frac{d}{dx} P_X(x) \qquad -\infty < x < \infty \qquad (9)$$

Equations 6 and 9 establish the relation between the pdf and cdf of a random variable.

We now present an example. Let X be the random variable defined previously, i.e., the peak acceleration at the center of gravity of an airplane during a flight with specified parameters. Let X have the pdf

$$p_X(x) = \begin{cases} 0.2e^{-0.2x} & x \geq 0 \\ 0 & x < 0 \end{cases} \qquad (10)$$

The graph of this function is shown in Figure 1. The cdf of X is obtained by integrating the pdf; it is

$$P_X(x) = \begin{cases} 1 - e^{-0.2x} & x > 0 \\ 0 & x < 0 \end{cases} \tag{11}$$

The graph of the cdf is shown in Figure 2. The probability of the event 4g < x ≤ 6g], is evaluated as follows:

$$P(4 < X < 6) = P_X(6) - P_X(4) = \left(1 - e^{-0.2(6)}\right)$$
$$- \left(1 - e^{-0.2(4)}\right) = 0.148 \tag{12}$$

b. Mean, variance, standard deviation--While the pdf and cdf of a random variable completely define the probabilistic character of the random variable, it is sometimes difficult to obtain an intuitive feeling for the values which a random variable is "most likely" to assume "on the average," by inspection of the mathematical functions. Therefore, we define the moments of a random variable. The fundamental moment is the mean, or expected value, and for a random variable, X, with pdf $p_X(x)$, this is defined

$$E[X] = \mu_X = \int_{-\infty}^{\infty} x p_X(x) dx \tag{13}$$

Clearly, this is the center of gravity of the pdf and defines the central value, or average value, of the random variable, X.

A moment which is an indicator of the dispersion of the realizations of the random variable is its variance. This is defined

$$V[X] = \sigma_X^2 = \int_{-\infty}^{\infty} \left(x - \mu_X\right)^2 p_X(x) dx \tag{14}$$

If the random variable, X, has a substantial probability mass concentrated away from its mean, then the variance will be relatively large. If the probability mass of X is concentrated closely about the mean, then the variance will be relatively small.

The standard deviation is defined as the square root of the variance.

$$\sigma_X = \sqrt{V[X]} \tag{15}$$

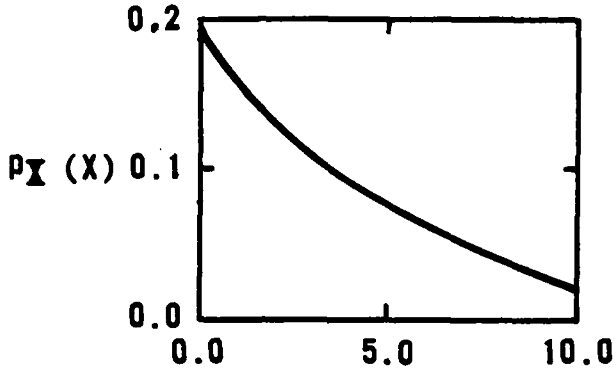

Figure 1. The pdf of the random variable, X.

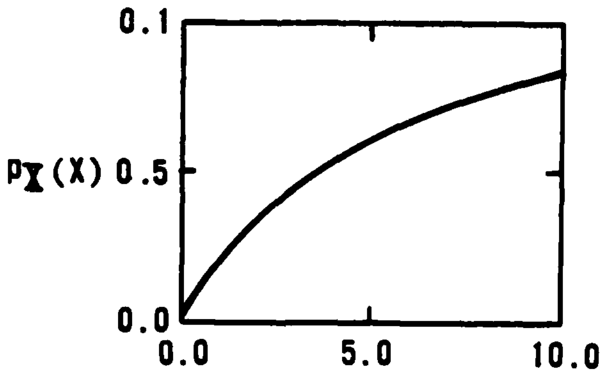

Figure 2. The cdf of the random variable, X.

This quantity has the same units as X and is a measure of the spread of the probability mass about the mean.

Finally, the mean square is a moment which measures the spread of probability mass about the origin. It relates to the variance and the mean as follows.

$$E[X^2] = V[X] + (E[X])^2 \tag{16}$$

From Equation 14, the variance is seen to be always nonnegative; therefore, $E[X^2]$ is always equal to or greater than the square of the mean.

For purposes of demonstration we find the moments of the random variable whose pdf is given in Equation 10. By using Equation 10 in Equations 13 and 14 we find

$$E[X] = 5 \tag{17a}$$

$$V[X] = 5 \tag{17b}$$

The standard deviation is

$$\sigma_X = \sqrt{5} \tag{17c}$$

These values concisely summarize the nature of the random variable defined previously.

c. Joint probabilities--The joint behavior of two or more random variables is often of interest. In problems of structural reliability analysis, consideration of multiple random variables is essential. We now proceed to outline the probability structure used to consider the joint behavior of random variables. As with single random variables, two functions can be used to describe the probabilistic character of a pair of random variables. These are the joint pdf and the joint cdf. The joint pdf for a pair of random variables, X and Y, is denoted $p_{XY}(x,y)$, and is defined in terms of the probabilities it yields for joint realizations of random variables. Where the symbol $\cap$ is used to denote the "and" operation we write

$$P(x_1 < X \le x_2 \cap y_1 < Y \le y_2) = \int_{x_1}^{x_2} dx \int_{y_1}^{y_2} dy \; p_{XY}(x,y) \tag{18}$$

This equation states that the probability that, as the result of a random experiment, the realization of X occupies the interval $(x_1, x_2]$, and the realization of Y occupies the interval $(y_1, y_2]$, is the volume under the surface $p_{XY}(x,y)$ over the two dimensional area defined by $(x_1, x_2]$ and $(y_1, y_2]$. Since a probability must be equal to or greater than zero, and a probability cannot be greater than one, we require that a joint pdf satisfy the following requirements.

$$p_{XY}(x,y) \geq 0 \qquad -\infty < x,y < \infty \tag{19a}$$

$$\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \; p_{XY}(x,y) = 1 \tag{19b}$$

The probability that the joint realization of X and Y falls within either of two nonoverlapping areas in the x-y plane is simply the sum of the probabilities that it falls within each area.

The joint pdf for several random variables is defined in a manner similar to that for two random variables. Let $X_1, X_2, \ldots X_n$, be jointly distributed random variables, and let their pdf be denoted $p_{X_1 X_2 \cdots X_n}(x_1, x_2, \ldots x_n)$; we can write

$$P\left(\alpha_{11} < X_1 \leq \alpha_{12}, \alpha_{21} < X_2 \leq \alpha_{22}, \ldots, \alpha_{n_1} < X_n \leq \alpha_{n_2}\right)$$

$$= \int_{\alpha_{11}}^{\alpha_{12}} dx_1 \int_{\alpha_{21}}^{\alpha_{22}} dx_2 \cdots \int_{\alpha_{n1}}^{\alpha_{n2}} dx_n \; p_{X_1 X_2 \cdots X_n}\left(x_1, x_2, \ldots x_n\right)$$

$$-\infty < \alpha_{i_1} \leq \alpha_{i_2} < \infty \tag{20}$$

The joint probability that each random variable, X_i, $i = 1, \ldots n$, is in the interval $\left(\alpha_{i_1}, \alpha_{i_2}\right]$, $i = 1, \ldots n$, is the integral of the joint pdf over the joint interval $\left(\alpha_{i_1}, \alpha_{i_2}\right)$, $i = 1, \ldots n$. This is a simple extension of the two-dimensional case. It is required that the joint pdf be nonnegative and have a unit integral over the infinite n-dimensional space (by analogy with Equation 19).

The joint pdf completely defines the probabilistic character of the random variables X and Y, and it can be used to obtain the marginal pdf's of X and Y. The marginal pdf of X is $p_X(x)$. The marginal pdf of X is obtained by integrating out dependence of $p_{XY}(x,y)$ on y. The marginal pdf of Y is found similarly. We write

$$p_X(x) = \int_{-\infty}^{\infty} p_{XY}(x,y)\, dy \qquad -\infty < x < \infty \tag{21a}$$

$$p_Y(y) = \int_{-\infty}^{\infty} p_{XY}(x,y)\, dx \qquad -\infty < y < \infty \tag{21b}$$

In general the joint pdf of X and Y cannot be obtained from the marginal pdf's.

The joint cdf can be used to characterize the joint probability behavior of a pair of random variables. The joint cdf of the random variables, X and Y, is denoted $P_{XY}(x,y)$, and is defined

$$P_{XY}(x,y) = \int_{-\infty}^{x} d\alpha \int_{-\infty}^{y} d\beta \, p_{XY}(\alpha,\beta), \; -\infty < x,y < \infty \tag{22}$$

This is the probability that the random variable X has a realization equal to or lower than x, and Y has a realization equal to or lower than y. Because of the requirements imposed on a pdf and the relation between a cdf and a pdf, we require that a cdf satisfy the following requirements.

$$P_{XY}(x,y) \geq 0 \qquad -\infty < x,y < \infty \tag{23a}$$

$$P_{XY}(-\infty, -\infty) = 0, \; P_{XY}(\infty,\infty) = 1 \tag{23b}$$

The joint cdf of several random variables is an extension of Equation 22. Let $X_1, X_2, \ldots X_n$, be jointly distributed random variables, and let their cdf be denoted $P_{X_1 X_2 \ldots X_n}(x_1, x_2, \ldots x_n)$. We define the cdf using the equation

$$P_{X_1 X_2 \ldots X_n}(x_1, x_2, \ldots, x_n)$$

$$= \int_{-\infty}^{x_1} d\alpha_1 \int_{-\infty}^{x_2} d\alpha_2 \ldots \int_{-\infty}^{x_n} d\alpha_n \, p_{X_1 X_2 \ldots X_n}(\alpha_1, \alpha_2, \ldots \alpha_n)$$

$$-\infty < x_i < \infty \tag{24}$$

This is the chance that each random variable, X_i, i = 1, ... n, has a realization in the interval $(-\infty, x_i)$, i = 1, ... n. The joint cdf is nonnegative, and assumes the value zero at $x_i = -\infty$, i = 1, ... n, and assumes the value one at $x_i = \infty$, i = 1, ... n. The marginal cdf's of X and Y can be obtained from the joint cdf of X and Y by evaluating the joint cdf as $x \to \infty$ or as $y \to \infty$. Specifically, we can write

$$P_X(x) = P_{XY}(x,\infty) \qquad -\infty < x < \infty \tag{25a}$$

$$P_Y(y) = P_{XY}(\infty,y) \qquad -\infty < y < \infty \tag{25b}$$

The first equation states that the probability that the realization of X is equal to or less than x and the realization of Y is finite is simply equal to the probability that the realization of X is equal to or less than x. The reason for this is that the realization of Y must be finite. The second equation is explained similarly.

Because of the manner in which the joint cdf is defined, it is possible to recover the joint pdf from it. By partial differentiation of the joint cdf with respect to the variables x and y, we obtain

$$p_{XY}(x,y) = \frac{\partial^2}{\partial x \partial y} P_{XY}(x,y), \; -\infty < x,y < \infty \tag{26}$$

d. Joint moments--Given an expression for the joint pdf of the random variables X and Y, we can obtain the individual moments of X and Y by first using Equations 21a and 21b to obtain the marginal pdf's of X and Y, and then using the moment definitions, Equations 13 through 16. In addition, there exist joint moments between X and Y which summarize, in brief, the probabilistic relation between the random variables. One joint moment is the correlation between X and Y; it is defined

$$E[XY] = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \; xy \; p_{XY}(x,y) \tag{27}$$

This is the average value of the product XY.

Another joint moment between X and Y is the covariance; it is defined

$$\begin{aligned} \mathrm{Cov}\,[X,Y] &= E\left[\left(X - \mu_X\right)\left(Y - \mu_Y\right)\right] \\ &= \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \; (x - \mu_X)(y - \mu_Y) \; p_{XY}\,(x,y) \end{aligned} \tag{28}$$

This is the average value of the product $(X - \mu_X)$ $(Y - \mu_Y)$ where μ_X and μ_Y are the means of X and Y, respectively. This is the lowest order joint movement of X and Y about their joint mean.

A third joint moment of X and Y is the correlation coefficient. It is defined

$$\rho_{XY} = \frac{\mathrm{Cov}[X,Y]}{\sigma_X \sigma_Y} \tag{29}$$

where σ_x and σ_y are the standard deviations of X and Y, respectively. This is simply a normalized form of the covariance. The correlation coefficient is a number which is always between -1 and 1. When the covariance or correlation coefficient is positive, this implies that the realizations of X and Y both tend to be on the same side of their means at the same time. We infer this since the product, $(X - \mu_X)\ (Y - \mu_Y)$, is "on the average" positive, and this implies that both factors are positive or both are negative. This is called positive correlation. A negative covariance or correlation coefficient implies that X and Y tend to be on opposite sides of their means at the same time. This is negative correlation. Finally, when the covariance or correlation coefficient is near zero, this indicates that the average of the product, $(X - \mu_X)\ (Y - \mu_Y)$, is near zero. Little or no linear relation between X and Y exists. The random variables are uncorrelated.

e. Independence--There is another condition, similar to correlation, which describes the relation between pairs of random variables. This is dependence. Two random variables are independent if their joint pdf is separable into two parts, one part dependent on x alone, and the other part dependent on y alone. That is, if

$$p_{XY}(x,y) = p_X(x)\ p_Y(y) \qquad -\infty < x,y < \infty \tag{30}$$

then the random variables, X and Y, are independent. Separability of the pdf implies separability of the cdf because the integral of Equation 18 can be separated. When X and Y are independent we have

$$P_{XY}(x,y) = P_X(x)\ P_Y(y) \qquad -\infty < x,y < \infty \tag{31}$$

When a pair of random variables is not independent, then it is dependent. Generally, random variables are independent when the realizations of one are not contingent in any way on the realizations of the other. Random variables are often assumed independent to simplify an analysis. Later, we discuss the reasons why a practical structural reliability analysis is considerably simplified when independence assumptions are used. When two random variables are independent, this implies that they are uncorrelated; but when two random variables are uncorrelated, this does not necessarily imply that they are independent.

Independence between pairs of several jointly distributed random variables is defined by simple extension of Equations 30 and 31. Let $X_1, X_2, \ldots X_n$, be jointly distributed random variables. The random variables are independent if their joint pdf can be written

$$p_{X_1 X_2 \ldots X_n}(x_1, x_2, \ldots, x_n) = p_{X_1}(x_1)\, p_{X_2}(x_2) \ldots p_{X_n}(x_n)$$
$$-\infty < x_i < \infty \tag{32}$$

and their joint cdf can be written

$$P_{X_1 X_2 \ldots X_n}(x_1, x_2, \ldots, x_n) = P_{X_1}(x_1)\, P_{X_2}(x_2) \ldots P_{X_n}(x_n)$$
$$-\infty < x_i < \infty \tag{33}$$

The concept of joint independence for several random variables is important in practical reliability analysis.

f. Analysis of reliability at one point--Given the background presented above, it is now possible to outline the steps in an elementary reliability analysis. The fundamental question that must be answered to determine the reliability of a structure at a single point is: What is the probability that, when the structure is loaded, the strength at the point will be greater than the load at the point? If we let X be a random variable denoting load at a point, and Y be a random variable denoting strength at the point, then we can define another random variable as

$$Z = Y - X \tag{34}$$

and this is simply the random margin between strength and load at the point. If we can find the probability that the realization of Z is equal to or greater than zero, then we will know the answer to the question posed above. To find the probability that the realization of Z is equal to or greater than zero, first develop an expression for the cdf of Z. Next, evaluate this at $z = 0$. Finally, subtract the result from 1. This yields the probability that the realization of Z is equal to or greater than zero.

The cdf of Z is developed in the following way.

Let $Z = Y - X$. The cdf of Z is, by definition

$$P_Z(z) = P(Z < z) \qquad -\infty < z < \infty \tag{35}$$

We can use the definition of Z on the right-hand side, above, to obtain

$$P_Z(z) = P(Y - X < z) = P(Y < z + X), \quad -\infty < z < \infty \tag{36}$$

In the final step the inequality is simply rearranged. The probability on the right-hand side is the integral of $p_{XY}(x,y)$ on the x-y plane, over those combinations of values of x and y such that $y < z + x$. Therefore, we can rewrite Equation 36 using either of the two equivalent expressions

$$P_Z(z) = \begin{cases} \int_{-\infty}^{\infty} dx \int_{-\infty}^{z+x} dy\; p_{XY}(x,y) & \text{(37a)} \\ \int_{-\infty}^{\infty} dy \int_{y-z}^{\infty} dx\; p_{XY}(x,y) & \text{(37b)} \end{cases} \quad -\infty < z < \infty$$

If, for example, the joint probability distribution of load X at a point, and strength Y at the same point, is known, then the cdf of the difference between those two can be obtained using the above integrals.

In many cases X and Y will be assumed independent (as in the case of load and strength) and Equations 37a and 37b will simplify to

$$P_Z(z) = \begin{cases} \int_{-\infty}^{\infty} p_X(x)\; P_Y(z + x)\; dx & \text{(38a)} \\ 1 - \int_{-\infty}^{\infty} p_Y(y)\; P_X(y - z)\; dy & \text{(38b)} \end{cases} \quad -\infty < z < \infty$$

These equations are much simpler to evaluate than Equations 37a and 37b, since these involve only a single integral. The reliability, R, or probability of survival at a point, is the probability that Z is equal to or greater than zero, and in terms of Equations 37a and 37b this is

$$R = \begin{cases} 1 - \int_{-\infty}^{\infty} dx \int_{-\infty}^{x} dy\; p_{XY}(x,y) & \text{(39a)} \\ 1 - \int_{-\infty}^{\infty} dy \int_{y}^{\infty} dx\; P_{XY}(x,y) & \text{(39b)} \end{cases}$$

When X and Y are independent, these equations simplify to

$$R = \begin{cases} 1 - \int_{-\infty}^{\infty} p_X(x)\; P_Y(x)\; dx & \text{(40a)} \\ \int_{-\infty}^{\infty} p_Y(y)\; P_X(y)\; dy & \text{(40b)} \end{cases}$$

A numerical example that uses a specific probability distribution is presented later.

The simplification introduced by assuming that the load and strength random variables are independent is made apparent in Equation 38. Beyond this simplification, though, lies another. Later, when we wish to consider the reliability of a complex structure, we will often assume that the event that a structure survives a load at one point (where the point load results from a complex external load) is independent of the event that the structure survives a load at another point. In this way, the overall structural reliability (probability of structural survival) can be computed as the product of point reliabilities (probabilities of structural survival at individual points).

We note that the pdf of $Z = Y - X$ can be obtained by differentiating Equation 37a or 37b with respect to z. We obtain

$$p_Z(z) = \begin{cases} \int_{-\infty}^{\infty} p_{XY}(x, z + x)\, dx & (41a) \\ & -\infty < z < \infty \\ \int_{-\infty}^{\infty} p_{XY}(y - z, y)\, dy & (41b) \end{cases}$$

When the random variables, X and Y, are independent, these equations simplify to

$$p_Z(z) = \begin{cases} \int_{-\infty}^{\infty} p_X(x)\; p_Y(z + x)\, dx & (42a) \\ & -\infty < z < \infty \\ \int_{-\infty}^{\infty} p_X(y - z)\; p_Y(y)\, dy & (42b) \end{cases}$$

These are convolution integrals and that fact can be used to simplify computations of the pdf of Z.

Another elementary approach to the solution of the fundamental reliability problem is also available. As before, let X be a random variable denoting load at a point on a structure, and let Y be a random variable denoting strength at the same point. In the present case, assume that X and Y can only assume values equal to or greater than zero; that is, the joint pdf of X and Y is zero when $x < 0$ and $y < 0$. This is a realistic assumption. We can define the ratio of these two random variables as another random variable W. It is

$$W = Y/X \qquad (43)$$

and this is the random factor of safety at the point on the structure. The probability that the realization of W, resulting from a random experiment, is equal to or greater than one is the structural reliability at the point. We can evaluate this by finding the cdf of W. By definition the cdf of W is

$$P_W(w) = P(W < w) \tag{44}$$

We use the definition of W to write

$$P_W(w) = P(Y/X < w) = P(Y < wX), \quad 0 < w < \infty \tag{45}$$

The last step is simply a rearrangement of the inequality. This is allowed since the realizations of X are nonnegative. The probability on the right-hand side is simply the integral of $p_{XY}(x,y)$ on the x-y plane over those combinations of values of x and y for which $y < wx$. Therefore, we can write two equivalent expressions for the cdf of W as

$$P_W(w) = \begin{cases} \int_0^\infty dx \int_0^{wx} dy \; p_{XY}(x,y) & (46a) \\ & 0 < w < \infty \\ \int_0^\infty dy \int_{y/w}^\infty dx \; p_{XY}(x,y) & (46b) \end{cases}$$

If the joint pdf of X and Y is known, then the cdf of W can be evaluated. When X and Y are independent, the integrals simplify to

$$P_W(w) = \begin{cases} \int_0^\infty p_X(x) \; P_Y(wx) \; dx & (47a) \\ & 0 < w < \infty \\ 1 - \int_0^\infty p_Y(y) \; P_X(y/w) \; dy & (47b) \end{cases}$$

These integrals are easier to evaluate than those in Equation 46a and 46b.

The reliability, R, at a point is the chance that the realization of W is equal to or greater than one. This is

$$R = \begin{cases} 1 - \int_0^\infty dx \int_0^x dy \; p_{XY}(x,y) & (48a) \\ 1 - \int_0^\infty dy \int_y^\infty dx \; p_{XY}(x,y) & (48b) \end{cases}$$

When X and Y are independent, these equations simplify to

$$R = \begin{cases} 1 - \int_0^\infty p_X(x)\ P_Y(x)\ dx & (48c) \\ \int_0^\infty p_Y(y)\ P_X(y)\ dy & (48d) \end{cases}$$

A numerical example using this expression is presented later.

g. The normal and lognormal distributions--We now discuss some specific probability distributions, namely the normal and lognormal distributions. These are important in structural reliability analysis. Let X and Y be random variables. We say that X and Y are jointly normal if their pdf is

$$p_{XY}(x,y) = \frac{1}{2\pi\ \sigma_X\ \sigma_Y \sqrt{1 - \rho_{XY}^2}} \cdot$$

$$\exp\left[-\frac{1}{2}\left(\left(\frac{x - \mu_X}{\sigma_X}\right)^2 - \frac{2\rho_{XY}(x - \mu_X)(y - \mu_Y)}{\sigma_X\ \sigma_Y} + \left(\frac{(y - \mu_Y)}{\sigma_Y}\right)^2\right)\right]$$

$$-\infty < x,y < \infty \tag{49}$$

The parameters of this distribution have the following meanings: μ_X and μ_Y are the means of X and Y; σ_X and σ_Y are the standard deviations of X and Y; and ρ_{XY} is the correlation coefficient between X and Y. When this joint pdf is used in Equation 21a, the marginal pdf of X can be obtained; it is

$$p_X(x) = \frac{1}{\sqrt{2\pi}\ \sigma_X} \exp\left[-\frac{1}{2}\left(\frac{x - \mu_X}{\sigma_X}\right)^2\right], \ -\infty < x < \infty \tag{50}$$

The pdf of Y is exactly analogous. Note that when ρ_{XY} equals zero in Equation 49 the joint pdf can be factored into two expressions as in Equation 50. This shows that when two normal random variables are uncorrelated they are also independent. This is not true for all probability distributions.

Figure 3 is a graph of the pdf of a normal random variable which has mean, $\mu_X = 0$, and variance, $\sigma_X^2 = 1$. Note that the pdf is symmetrical and has a finite value for every value of x.

It is possible for a large collection of random variables to be normally distributed. Let X_1, X_2, ... X_n, be a collection of random variables which is jointly normally distributed. Their pdf is given by

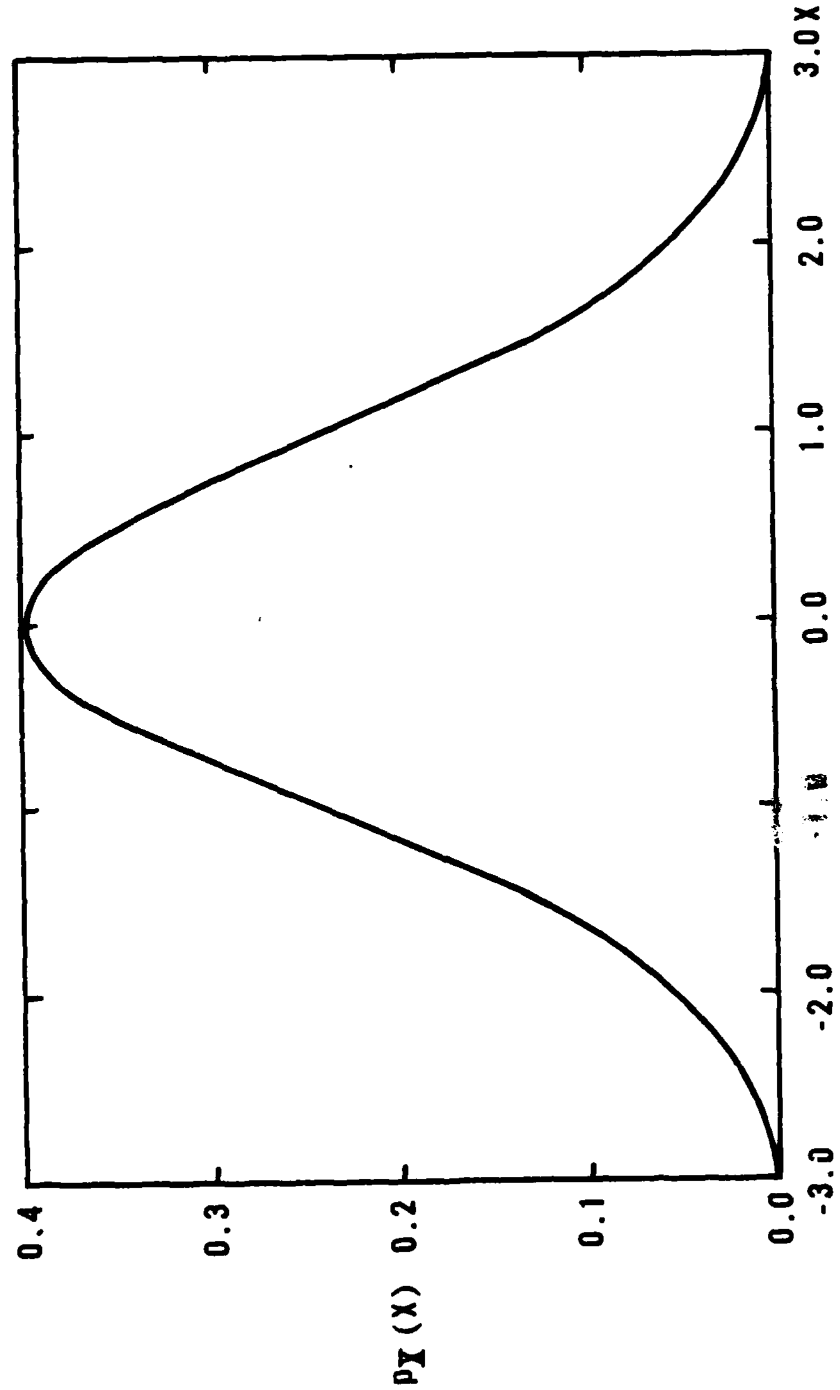

Figure 3. The pdf of a normally distributed random variable. $\mu_X = 0$, $\sigma_X^2 = 1$.

$$p_{X_1 X_2 \ldots X_n}(x_1, x_2, \ldots x_n)$$

$$= \frac{1}{(2\pi)^{n/2} \, |S|^{1/2}} \exp\left[-\frac{1}{2}(\{x\} - \{\mu\})^T [S]^{-1} (\{x\} - \{\mu\})\right] \tag{51a}$$

$$-\{\infty\} < \{x\} < \{\infty\}$$

where $\{x\}$ is a vector of x_i values defined

$$\{x\} = (x_1 \; x_2 \; \ldots \; x_n)^T \tag{51b}$$

$\{\mu\}$ and [S] are the mean vector and covariance matrix, defined

$$\{\mu\} = (\mu_1 \; \mu_2 \; \ldots \; \mu_n)^T \tag{51c}$$

where

$$\mu_i = E[X_i], \; i = 1,2, \ldots n \tag{51d}$$

and

$$[S] = \begin{bmatrix} S_{11} & S_{12} & \cdots & S_{1n} \\ S_{21} & S_{22} & \cdots & S_{2n} \\ \vdots & & & \\ S_{n1} & S_{n2} & \cdots & S_{nn} \end{bmatrix}$$

where

$$S_{ij} = \mathrm{Cov}\,[X_i, X_j], \; i,j = 1,2, \ldots n$$

$|S|$ is the determinant of the matrix, [S]. A superscript, T, refers to the matrix operation of transposition. A superscript, -1, refers to inversion.

One of the major difficulties in the use of the normal distribution is that it cannot be integrated, except numerically. We denote the cdf of a standardized normal random variable as $\Phi(x)$, and this function is tabulated in many references (e.g., Refs. 1 and 2). This is the cdf of a normal random variable

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1. Brownlee, K. A., Statistical Theory and Methodology in Science and Engineering, John Wiley and Sons, Inc., New York, 1960.

2. Abramowitz, M. and Stegun, I. A., eds., Handbook of Mathematical Functions, National Bureau of Standards, Applied Math Series 55, June, 1964.

with mean zero and variance 1. The $\Phi(x)$ is related to the error function by the formula

$$\Phi(x) = \frac{1}{2}\,(1 + \text{erf}\,(x/\sqrt{2})),\ -\infty < x < \infty \tag{52}$$

The error function is a function available on most computers; therefore, in numerical analyses the cdf of a normal random variable can be easily evaluated. Through the use of a simple linear transformation, the cdf of a normal random variable with nonzero mean and nonunit variance can be obtained from $\Phi(x)$. Let X be a normally distributed random variable with mean, μ_X, and variance σ_X^2; the cdf of X is

$$P_X(x) = \Phi\left(\frac{x - \mu_X}{\sigma_X}\right), \quad -\infty < x < \infty \tag{53}$$

When X and Y are independent normal random variables, their joint cdf can be expressed using $\Phi(x)$. Let the means and variances of X and Y be μ_X and μ_Y and σ_X^2 and σ_Y^2, respectively. The joint cdf and X and Y is

$$P_{XY}(x,y) = \Phi\left(\frac{x - \mu_X}{\sigma_X}\right)\Phi\left(\frac{y - \mu_Y}{\sigma_Y}\right), \quad -\infty < x,y < \infty \tag{54}$$

The joint cdf of correlated normal random variables can be expressed only by writing the double integral of Equation 22 where $p_{XY}(x,y)$, is given in Equation 49 and ρ_{XY} is not zero. In the normal case a numerical approach must be used to evaluate $P_{XY}(x,y)$ when ρ_{XY} is nonzero.

Pairs of normally distributed random variables possess the useful future that their sums and differences are also normally distributed. Let X and Y be normal random variables with means and variances, μ_X and μ_Y, and σ_X^2 and σ_Y^2, respectively. And let their correlation coefficient be ρ_{XY}. Then the random variable

$$Z = Y - X \tag{55a}$$

is a normally distributed with mean and variance

$$E[Z] = \mu_Z = \mu_Y - \mu_X \tag{55b}$$

and

$$V[Z] = \sigma_Z^2 = \sigma_Y^2 + \sigma_X^2 - 2\rho_{XY}\sigma_X\sigma_Y \tag{55c}$$

The random variable, Z, defined above, is very useful in the analysis of structural reliability at a point. Let us identify the random variable, Y, with structural material strength at a point, and the random variable, X, with load at the same point. Then the reliability at that point is the probability that Z is equal to or geater than zero. When X and Y are normal, Z is normal and the point reliability, R, can be expressed

$$R = P(Z > 0) = 1 - P(Z < 0)$$
$$= 1 - \Phi\left(-\mu_Z/\sigma_Z\right) \tag{56}$$

When the load and strength random variables are independent this expression can be simplified using Equation 55.

$$R = 1 - \Phi\left(\frac{\mu_X - \mu_Y}{\sqrt{\sigma_X^2 + \sigma_Y^2}}\right) = \Phi\left(\frac{\mu_Y - \mu_X}{\sqrt{\sigma_X^2 + \sigma_Y^2}}\right) \tag{57}$$

The final step on the right can be made because of the symmetry of the normal pdf. This equation is used often for reliability analysis.

In reliability analysis it is often desirable to consider the reliability at a point on a structure in relation to the factor of safety of the classical design at that point. This can be done in the following way. Let X be a random variable representing load at a point, and let Y represent strength. Assume that these are independent. Let these random variables be distributed normally, with means μ_X and μ_Y and variances σ_X^2 and σ_Y^2. Assume that the allowable load, x_0, in the classical analysis is chosen so that a random load has a probability, p_x, of being lower than, x_0; we can write

$$p_x = \Phi\left(\frac{x_0 - \mu_X}{\sigma_X}\right) \tag{58a}$$

Assume that the design strength, y_0, in the classical analysis is chosen so that a random strength has a probability, p_y, of exceeding y_0; we can write

$$p_y = 1 - \Phi\left(\frac{y_0 - \mu_Y}{\sigma_Y}\right) \tag{58b}$$

The above two equations can be inverted to obtain x_0 and y_0.

$$x_0 = \sigma_X\, \Phi^{-1}(p_x) + \mu_X \tag{58c}$$

$$y_0 = \sigma_Y\, \Phi^{-1}(1 - p_y) + \mu_Y \tag{58d}$$

where $\Phi^{-1}(.)$ is the inverse function of the standard normal cdf. The factor of safety, F.S., is simply the ratio of y_0 to x_0.

$$F.S. = \frac{y_0}{x_0} = \frac{\sigma_Y\Phi^{-1}(1 - p_y) + \mu_Y}{\sigma_X\Phi^{-1}(p_X) + \mu_X} \tag{58e}$$

The point reliability for this case is given by Equation 57. More general formulas for accomplishing this comparison are given in Reference 3.

The use of the normal distribution is demonstrated by the following example. Suppose that a structure is to be subjected to a random environment, and if failure occurs it is most likely to occur at a single predictable point. Assume that failure occurs when material yielding occurs. The environment is random, and the probability distribution of the peak load random variable, X, at the point of potential failure is normal. The mean and variance of the peak load are 1.5×10^8 Pa and 9.0×10^{14} $(Pa)^2$, respectively. The strength, Y, of the component is a random variable, also, with mean and variance 4.0×10^8 Pa and 16.0×10^{14} $(Pa)^2$, respectively. This random variable is also normally distributed and is independent of X. Further, we assume that the structural component under consideration was designed deterministically, and its allowable load is 2.1×10^8 Pa and its design strength is 3.2×10^8 Pa. Then the margin of safety for the element is 0.5; its factor of safety is 1.5. Find (1) the probability that the allowable load is surpassed in the test, (2) the probability that the actual strength is lower than the design strength, and (3) the structural reliability.

The probability that the allowable load is surpassed during one random experiment is the chance that a realization of X is equal to or greater than 2.1 $\times 10^8$ Pa. Using Equation 53 and Table 1 in Reference 1, we find that this is

$$P(X > 2.1 \times 10^8) = 1 - P(X < 2.1 \times 10^8) = 1 - \Phi(2) = 0.02275 \tag{59a}$$

The probability that the realized yield strength is lower than the design yield strength is the chance that a realization of Y is equal to or less than 3.2 $\times$ 10^8 Pa. Using Equation 53 and Table 1 in Reference 1, we find that this is

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3. Merchant, D. H., et al., Study of Ground Handling Equipment Design Factors, AFWL-TR-77-150, Air Force Weapons Laboratory, Kirtland AFB, NM, February 1978.

$$P(Y < 3.2 \times 10^8) = \Phi(-2) = 0.02275 \tag{59b}$$

Let Z = Y - X. The moments of Z are

$$\mu_Z = 2.5 \times 10^8 \text{ Pa} \tag{59c}$$

$$\sigma_Z^2 = 25.0 \times 10^{14} \text{ (Pa)}^2 \tag{59d}$$

and Z is normally distributed. Failure occurs when Z is less than zero, so the reliability is the probability that Z is equal to or greater than zero. This is

$$R = P(Z > 0) = 1 - P(Z < 0) = 1 - \Phi(-5.0) = 1 - (0.30 \times 10^{-6}) \tag{60}$$

This important quantity is the chance of structural survival during one random experiment.

In describing this numerical example, it was stated that failure is assumed to occur with yielding. Since it is relatively easy to analyze linear structural response, it would not be difficult to determine what responses are caused by specific loads on a structure. For the most complicated structures, finite element analyses could be used. It might alternately have been stated that failure occurs at some stress beyond the yield level. In this case, the analyst could determine the structural reliability as it was done above, but the nonlinear responses caused by structural inputs must be determined. This implies that a nonlinear structural analysis must be performed, and this adds a degree of complexity to the problem.

Another probability distribution important in structural reliability analysis is the lognormal distribution. A random variable is lognormally distributed when its logarithm is normally distributed. Let X be a lognormally distributed random variable and let

$$U = \ln X \tag{61}$$

Then U is a normally distributed random variable. If X has mean μ_X and variance σ_X^2, then the moments of the random variable, U, are

$$E[U] = \mu_U = \ln\left(\mu_X/\sqrt{V_X^2 + 1}\right) \tag{62a}$$

$$V[U] = \sigma_U^2 = \ln\left(V_X^2 + 1\right) \tag{62b}$$

Where V_X is the coefficient of variation of the random variable X, and is defined

$$V_X = \sigma_X/\mu_X \tag{62c}$$

The cdf of the random variable X is defined as follows.

$$P_X(x) = P(X < x) = P(e^U < x) = P(U < \ell nx)$$

$$= \Phi\left(\frac{\ell nx - \mu_U}{\sigma_U}\right), \; x > 0 \tag{63}$$

This shows that the cdf of a lognormal random variable can be written in terms of the cdf of a standard normal random variable, $\Phi(.)$.

The pdf of a lognormally distributed random variable can be obtained by differentiating Equation 63 with respect to x. The graph of a lognormal cdf is shown in Figure 4, for a random variable with mean, $\mu_X = 10$, and variance, $\sigma_X^2 = 1$. Note that the pdf is nonzero only on the right half of the real line. The distribution is skewed toward the right.

The products and quotients of a pair of lognormally distributed random variables possess the useful feature that they are also lognormally distributed. The reason for this is that the logarithm of a product is the sum of the logarithms of the multiplicands, and the logarithm of a quotient is the difference between the logarithms of the factors. Further, the logarithm of each factor has a normal distribution, and sums and differences of normal random variables are also normally distributed.

To see how this can be useful in reliability computations, consider the following. Let Y be a lognormally distributed random variable denoting strength at a point on a structure, and let X be a lognormally distributed random variable denoting peak load at the same point. Assume that these are independent. Let W be defined

$$W = Y/X \tag{64}$$

The structural reliability is the probability that a realization of W is equal to or greater than 1. Let us also define

$$U_1 = \ell n\; X \tag{65a}$$

$$U_2 = \ell n\; Y \tag{65b}$$

The moments of U_1 and U_2 are

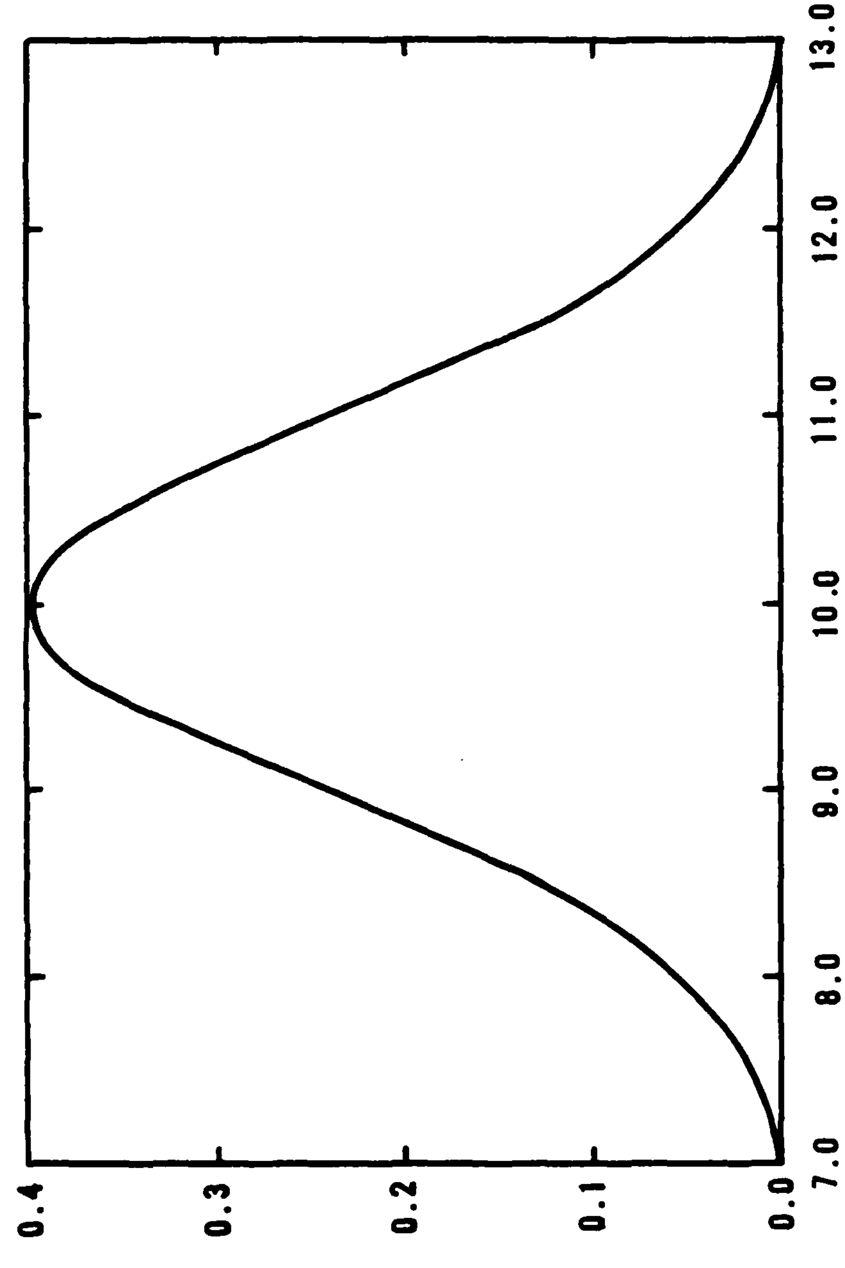

Figure 4. The pdf of a lognormally distributed random variable. $\mu_X = 10$, $\sigma_X^2 = 1$.

$$\mu_{U_1} = \ln\left(\mu_X/\sqrt{V_X^2 + 1}\right) \tag{66a}$$

$$\mu_{U_2} = \ln\left(\mu_Y/\sqrt{V_Y^2 + 1}\right) \tag{66b}$$

$$\sigma_{U_1}^{\ 2} = \ln\left(V_X^2 + 1\right) \tag{66c}$$

$$\sigma_{U_2}^{\ 2} = \ln\left(V_Y^2 + 1\right) \tag{66d}$$

The reliability is computed as follows

$$R = P(W > 1) = 1 - P(W < 1) = 1 - P(Y/X < 1)$$

$$= 1 - P(\ln Y - \ln X < \ln 1) = 1 - P(U_2 - U_1 < 0)$$

$$= 1 - \Phi\left(\frac{\mu_{U_1} - \mu_{U_2}}{\sqrt{\sigma_{U_1}^{\ 2} + \sigma_{U_2}^{\ 2}}}\right) \tag{67}$$

The reliability of a structure whose load and strength are lognormally distributed can be expressed in terms of the cdf of a standard normal random variable, $\Phi(.)$.

To demonstrate the use of this approach, repeat the numerical example solved previously. In the present case, though, assume that the load and strength are lognormally distributed. The lognormally distributed random variable, X, which represents peak load at a point on the structure, has mean 1.5×10^8 Pa, and variance 9.0×10^{14} $(\text{Pa})^2$. The lognormally distributed random variable, Y, which represents the strength of the structure, has mean 4.0×10^8 Pa, and variance 16.0×10^{14} $(\text{Pa})^2$. In the deterministic structural design, the allowable load was 2.1×10^8 Pa, and the design was 3.2×10^8 Pa. These yield a safety margin of 0.5, or a factor of safety of 1.5. We wish to find (1) the probability that the actual load is surpassed in the test, (2) the probability that the actual strength is lower than the design strength, and (3) the structural reliability.

The probability that the allowable load is surpassed during loading is the chance that a realization of X is equal to or greater than 2.1×10^8 Pa. Using Equations 62 and 63 and Table 1 in Reference 1, this can be evaluated.

$$P(X > 2.1 \times 10^8) = 1 - \Phi(1.77) = 0.0384 \tag{68}$$

The probability that the realized yield stress is lower than the design yield stress is the chance that realization of Y is equal to or less than 3.2×10^8 Pa. Using Equations 62 and 63 and Table 1 in Reference 1, this can be evaluated.

$$P(Y < 3.2 \times 10^8) = \Phi(-2.51) = 0.00604 \tag{69}$$

Finally, using the variables U_1 and U_2, defined in Equations 65 we can compute their moments. They are

$$\mu_{U_1} = 18.81 \tag{70a}$$

$$\mu_{U_2} = 19.80 \tag{70b}$$

$$\sigma_{U_1}^2 = 0.0392 \tag{70c}$$

$$\sigma_{U_2}^2 = 0.00995 \tag{70d}$$

Equation 67 can be used to evaluate the structural reliability at a point. It is

$$R = 1 - (0.40 \times 10^{-5}) \tag{71}$$

This is an estimate of the chance of structural survival during the random experiment.

The results of the numerical examples ending with Equations 60 and 71 provide a comparison between structural reliability estimates obtained using the normal assumption for the load and strength random variables, and the lognormal assumption for the load and strength random variables. If we recall that the probability of failure of a system is given by 1 minus the reliability, it is clear that the lognormal assumption provides a more conservative estimate of failure probability than the normal assumption. If the analyst wishes to be conservative, then, based on this example, he will choose the lognormal assumption over the normal assumption regardless of which assumption more closely approximates the true state of nature. It happens in most cases of practical interest that it is difficult to show that one distribution is better suited for use in a particular problem than another, especially when the distributions are similarly shaped. In spite of this, many analysts favor the lognormal distribution over the normal because the lognormal distribution is nonzero only over the nonnegative half of the real number line, whereas the normal distribution is nonzero over the entire real number line. In practice, quantities such as peak load and failure strength must be positive; therefore, the lognormal distribution is thought to be a more reasonable model for their probabilistic behavior. This reasoning leads to a conclusion of secondary importance, though; in fact, the model which a probability distribution provides for a natural phenomenon should be well fitted everywhere. Moreover, there is no guarantee that the right-hand tail of a lognormal distribution will describe a practical phenomenon better than the right-hand tail of a normal distribution. We conclude from this that use of the normal distribution in practical problems is quite acceptable.

Another reason why the normal distribution may be used in practical problems is that the results of a reliability analysis are made conservative by other assumptions in the analysis. For example, the often used assumption that failure occurs when the yield stress is surpassed tends to make reliability analyses conservative if true failure occurs when the response is beyond the yield point. The assumption of independence, to be discussed later, also makes analyses conservative.

Note that, whether the normal or lognormal assumption is used, the estimated probability of failure is relatively small. This result takes place even though the factor of safety, design strength and allowable load are quite modest. It should be noted that the results are strongly dependent on the random variable moments. In general, as the variances of the load and strength increase, the reliability tends to diminish.

A summary of the information and formulas presented above is given in the last section in this chapter.

2. RANDOM PROCESS CONCEPTS

In the performance of a practical reliability analysis it is possible that the analyst will be provided with the type of information used in the previous section. That is, the analyst may be provided with the probabilistic character of the structural material and the peak load to be applied on a structure during a random experiment. In fact, some materials have been probabilistically characterized and their parameters tabulated; these will be discussed later. It is more likely, however, that it will be necessary to derive the probability distribution of peak loading from some other information which is available from field measurements.

It is most likely that the loads which will be applied to a structure during its design life will be either dynamic, or multiple repetitions of a static load. This report assumes that the strength of each structure under consideration does not deteriorate. Therefore, we wish to find the probability distribution of the highest peak load in a random dynamic input, or the largest load in a sequence of static inputs.

This section discusses the procedure which can be used to obtain the peak load probability distribution using the parameters of a random input. Particularly, consider the case where the input is a random process and focus most attention on the stationary random process.

Prior to determining the peak load probability distribution, some elementary concepts in random processes are briefly introduced.

a. Random processes--Formally, a random process is defined as a parametered family of random variables. In other words, a random process can be thought of as a collection of random variables unfolding in some sequence, such as a time sequence. Random processes are used to describe measures of interest in random structural dynamics problems. For example, the pressure at a point on the outer surface of a re-entry vehicle passing through the atmosphere is a random process. And, for example, the stress at a point on a bomb rack attaching a bomb to an airplane flying through the atmosphere is a random process.

Random processes can be divided into those which vary rapidly with time and those which do not. A random process varies rapidly with time when some of its characteristics, such as total instantaneous average power, or instantaneous average power in a limited band of frequencies, show considerable variation on a time scale which is on the order of, or shorter than, the fundamental period of the mechanical system under consideration. Therefore, the rapidity of variation of a random process depends on the system which the random process is used to excite. Random processes which vary rapidly with time require special treatment which will be discussed at the end of this section. The other class of random processes is known as stationary or quasi-stationary random processes. These random processes have a steady state character in a random sense.

Random processes are completely specified by collections of joint pdf's which define the probabilistic relations among every collection of random variables composing the random process. When we assume that the random process under consideration is stationary and normally distributed, though, it is not necessary to directly consider all these pdf's and some meaningful results can be obtained using relatively simple calculations. A normal random process is one whose joint pdf's are all normal in form (see Ref. 6). A normal random process is characterized by its mean, its variance, and the correlation between each pair of random variables in the random process. (Most input and response random processes are assumed normal, in practice.) For a stationary normal random process, many quantities of interest can be determined using the mean of the random process, which we assume constant in time, and a function known as the spectral density.

The spectral density of a stationary random process is a function which defines the mean square power of a random process in the frequency domain. This relates directly to the behavior of the random process in the time domain, and the peak values that a random process executes can be approximately characterized using the spectral density.

We now define spectral density and show how it is computed. Let $X(t)$ be a stationary normal random process with zero mean. (If the mean of a random process is nonzero, then the mean must be subtracted out and considered separately from the oscillatory portion of the random process.) Let $G(f)$ represent the spectral density of the random process. From the random process, $X(t)$, we generate another random process which is its finite Fourier transform.

$$X(f,T) = \int_0^T X(t)e^{-i2\pi ft}dt, \quad f > o \tag{72}$$

From this random process we generate another which is the squared modulus of $X(f,T)$; it is

$$|X(f,T)|^2 = X(f,T) \cdot X^*(f,T), \quad f > o, \; T > 0 \tag{73}$$

where a star superscript refers to the operation of complex conjugation. Finally, take the mean of this random process and multiply by $2/T$. The limit of this quantity as T approaches infinity is the spectral density of $X(t)$.

$$G_X(f) = \lim_{T\to\infty} \frac{2}{T} E[\,|X(f,T)|^2], \quad f > 0 \tag{74}$$

This spectral density of X(t) defines the density of mean square power at the frequency, f.

This formula is commonly used to estimate the spectral density. Yet two things about the formula create difficulties in this regard. First, we will never have a record with infinite length; second, we cannot even estimate the expected value unless several records are available to average. The following procedure is used.

A signal which comes from a random source is called a realization of the random process at the source. When one realization, x(t), is to be used to estimate the spectral density, operate as follows. Divide the signal, x(t), into N parts, each of which is denoted $x_j(t)$, j = 1, ... N. Then use each part, $x_j(t)$, in Equations 72 and 73 to obtain the estimate of Equation 74; this we denote $G_{X_j}(f)$. Finally, we average the $G_{X_j}(f)$ over all j = 1, ... N, at each frequency to obtain our estimate, $\hat{G}$ (f). The fact that T is finite leads to an estimate of $G_X(f)$ rather than an exact computation of the true underlying value. The fact that only one signal is available is handled by dividing the available signal into parts, as shown above. It should be noted that the single signal, x(t), used to estimate the spectral density must be representative of all others in the random process. This quality in a random process is called ergodicity.

We note that the area under the spectral density curve is the variance of the random process, X(t).

$$V[X(t)] = \sigma_X^2 = \int_0^\infty G_X(f)\, df \tag{75}$$

For a stationary random process this quantity is a constant. The positive square root of the random process variance is its standard deviation.

The average behavior of a random process in the time domain can be characterized in terms of the random process spectral density. When the spectral density has values in a relatively small frequency range which are larger than the other spectral density values at other frequencies, then the power in the random process tends to be concentrated in the small frequency range. For example, if the spectral density of a random process appears as shown in Figure 5a, then the realizations of the random process will appear as shown in Figure 5b. That is, when the spectral density of a random process shows that the mean square power is concentrated in one frequency range near f_n, then the random process realizations tend to look like harmonic signals with frequency, f_n, and random amplitude. When the spectral density shows concentrations of power at two frequencies, then the realizations of the random process tend to look like the superposition of two signals, such as that shown in Figure 5b, at the two frequencies of interest.

At the extreme, when the spectral density of a stationary random process is nearly constant, the realizations of the random process tend to look like signals composed of many components at many frequencies. For example, if the spectral density of a stationary random process is given by the graph in Figure 6a, then the realizations of the random process may resemble the signal shown in Figure 6b.

b. Distribution of the largest peak in a stationary random process--To date, the exact probability distribution of the largest peak value in a random signal of interest in mechanical analysis has not been obtained. However, approximate probability distributions of the largest peak value in a stationary random process can be obtained in a number of ways. Three of these are described in the following paragraphs.

It can be shown that, when a normal random process is sampled a large number of times, the largest value in the sample is approximately governed by a Type I extreme value distribution. The parameters of the extreme value distribution depend on the time duration over which the random process is sampled, and the spectral density of the sampled random process.

In the following, consider a mean zero, normal random process, $X(t)$, with spectral density, $G_X(f)$, and variance, σ_X^2. The modifications that are required in the following formulas in the cases where the mean of $X(t)$ is not zero are discussed later. Let Z be the largest, observed, normalized value in $X(t)$, when the random process is observed during the time $(0,T)$. Z is given by

$$Z = \max_{(0,T)} \frac{X(t)}{\sigma_X} \tag{76}$$

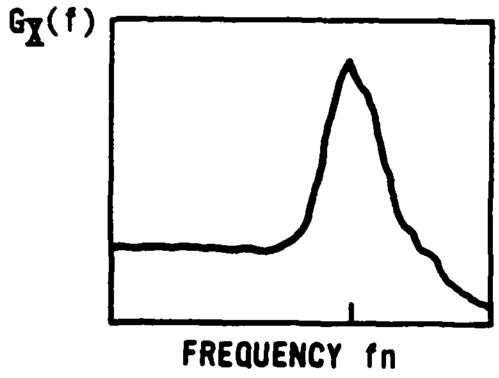

Figure 5a. Spectral density of a narrowband random process.

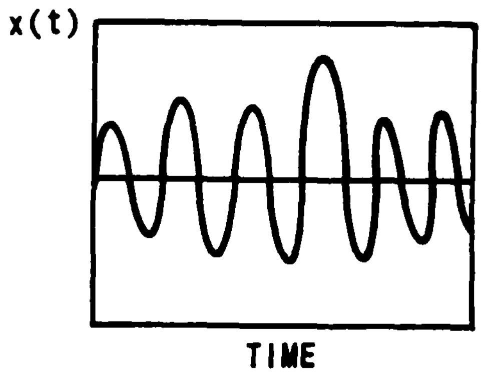

Figure 5b. Realization of a narrowband random process.

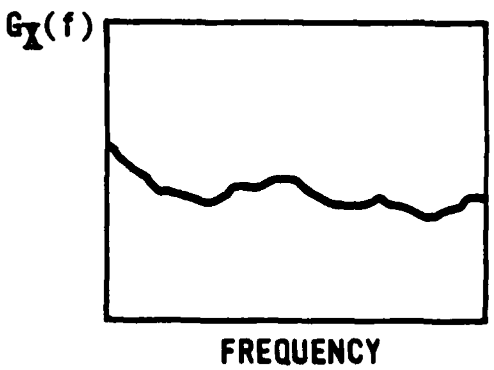

Figure 6a. Spectral density of a wideband random process.

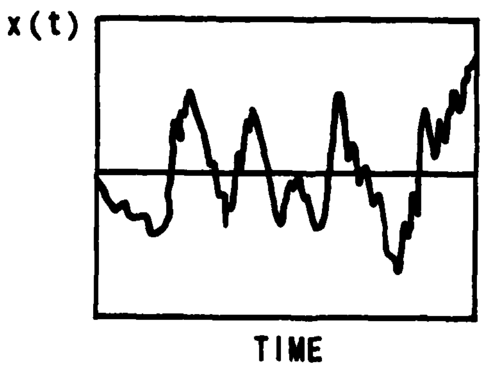

Figure 6b. Realization of a wideband random process.

The distribution of Z is approximately a Type I extreme value distribution; this is

$$P_Z(z) = \exp\left[-e^{-\alpha(z-u)}\right], \quad -\infty < z < \infty \tag{77}$$

where u is known as the characteristic largest value of Z, and α is the extremal intensity function. These will be defined shortly in terms of the parameters of X(t).

The mean value and variance of Z are given by

$$\mu_Z = u + \frac{0.5772}{\alpha} \tag{78a}$$

$$\sigma_Z^2 = \frac{\pi^2}{6\alpha^2} \tag{78b}$$

These are obtained when the derivative of Equation 77 is used in Equations 13 and 14.

The parameter u depends on the duration of the stationary random process input during which the maximum will occur, and on the random process spectral density. We define the expected number of times that a stationary random process with zero mean will cross the value zero with positive slope per unit time. This is (Ref. 4)

$$E[N_+(0)] = \left[\frac{\int_0^\infty f^2 G_X(f)\, df}{\int_0^\infty G_X(f)\, df}\right]^{1/2} \tag{79}$$

Let T be the time duration during which the maximum can occur. Then the normalized characteristic largest value is

$$u = \left[2 \ln \left(T \cdot E[N_+(0)]\right)\right]^{1/2} \tag{80}$$

Since we have assumed that the underlying random process, X(t), is normal, we can express the extremal intensity function, α, in terms of u as (Ref. 5).

$$\alpha = u + u^{-1} - 2u^{-3} + 10u^{-5} \tag{81}$$

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4. Rice, S. O., "Mathematical Analysis of Random Noise," Bell System Technical Journal, V. 23, 24, Reprinted in: Selected Papers on Noise and Stochastic Processes, Ed. Wax, Nelson, Dover Publications, Inc., New York, 1954.
5. Gumbel, E. J., Statistics of Extremes, Columbia University Press, New York, 1958.

After computing $E[N_+(0)]$ from Equation 79, u can be computed from Equation 80, and then α can be computed from Equation 81. The u and α can then be used in Equation 78 to find the mean and variance of the largest value in the random process, and they can be used in the cdf of Z, Equation 77.

When the mean value of the load random process, X(t), is not zero then the cdf of the normalized highest peak in the load, Z, must be modified to reflect this fact. Let the mean of the load random process be μ_X. Then we are interested in the probability that the peak load is less than z units beyond the normalized mean, μ_X/σ_X. This is

$$P\left(Z < z - \frac{\mu_X}{\sigma_X}\right) = \exp\left[-e^{-\alpha\left(z-\left(u + \mu_X/\sigma_X\right)\right)}\right], \quad -\infty < z < \infty \tag{82}$$

Note that when μ_X is zero, this expression is identical to Equation 77. The mean value of the normalized peak must be modified, too. This becomes

$$\mu_Z + \frac{\mu_X}{\sigma_X} \tag{83}$$

The variance of the highest peak is not affected by the shift in mean.

Now consider a practical example. Suppose that an element is loaded in such a way that the mean load on the element is $\mu_X = 7.0 \times 10^7$ Pa, and the spectral density of the load (excluding the effects of the mean) is given by the graph in Figure 7. The load is applied for a duration of 60 seconds. Let Z be the random variable representing the greatest load on the element during the 60 seconds when the load is applied. Find (1) the parameters of the probability distribution governing the peak load, Z, (2) the mean and variance of Z, and (3) write the cdf of Z. The variance of the underlying load random process, X(t), is

$$\sigma_X^2 = \int_0^\infty G_X(f)\, df = 1.96 \times 10^{14} (\text{Pa})^2 \tag{84}$$

The expected zero crossing rate of the random process is

$$\begin{aligned} E[N_+(0)] &= \left[(1.96 \times 10^{14})^{-1} \int_0^\infty f^2 G_X(f)\, df\right]^{1/2} \\ &= 964 \text{ Hz} \end{aligned} \tag{85}$$

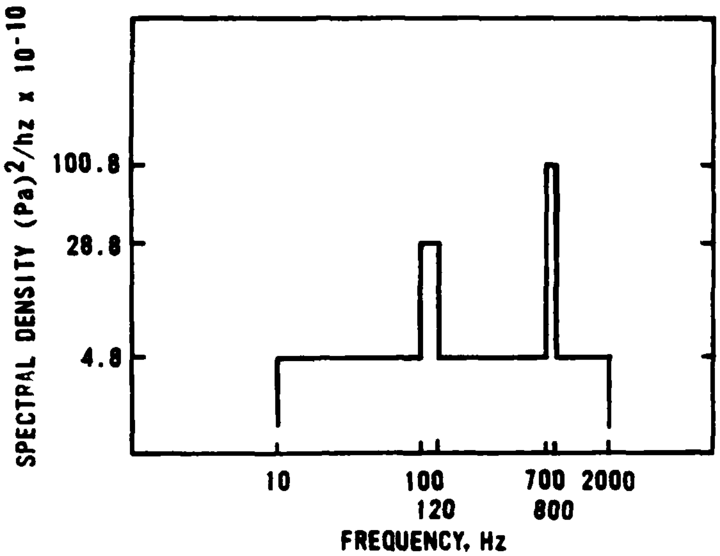

Figure 7. Pressure spectral density for example problem.

The normalized characteristic largest value is

$$u = [2 \ell n \ (T \cdot E[N_+(0)])]^{1/2} = 4.69 \tag{86}$$

The extremal intensity function is

$$\alpha = u + u^{-1} - 2u^{-3} + 10u^{-5} = 4.89 \tag{87}$$

The parameters u and α can be used to find the mean and variance of Z. These are

$$\mu_Z = u + \frac{0.5772}{\alpha} = 4.81 \tag{88}$$

$$\sigma_Z^2 = \frac{\pi^2}{6\alpha^2} = 0.069 \tag{89}$$

The above computations were performed ignoring the fact that the present random process has a nonzero mean. Therefore, the mean value of Z, μ_Z, computed above, is simply the average of the highest peak in the random process above the underlying mean, μ_X, which in the present case is 7.0×10^7 Pa. From Equation 83, the true mean of the normalized peak is given by

$$\mu_Z + \frac{\mu_X}{\sigma_X} = 9.81 \tag{90}$$

The nonnormalized mean value of the highest peak load can be obtained by simply multiplying the expression in Equation 83 by σ_X; this yields

$$\mu_P = \mu_Z\sigma_X + \mu_X \tag{91a}$$

The nonnormalized variance of the highest peak load is obtained by multiplying the normalized variance by $\sigma_X{}^2$.

$$\sigma_P{}^2 = \sigma_X{}^2\sigma_Z{}^2 \tag{91b}$$

In the present numerical example these are

$$\mu_P = 13.7 \times 10^7 \text{ Pa} \tag{92a}$$

$$\sigma_P{}^2 = 1.35 \times 10^{13} \text{ (Pa)}^2 \tag{92b}$$

Finally, the cdf of the normalized highest peak load is the probability that a realization of Z exceeds the normalized underlying mean, μ_X/σ_X by an amount equal to or less than z.

In the present numerical example this is

$$P\left(Z \leq z - \mu_X\right) = \exp\left[-\, e^{-4.89(z - 9.69)}\right]$$
$$-\infty < z < \infty \tag{93}$$

This cdf is plotted in Figure 8.

The cdf of the actual peak can be obtained using Equation 76. If we define Y as the actual highest peak in the random process we have

$$Y = \max_{(0,T)} X(t) \tag{94}$$

The cdf of Y is

$$P_Y(y) = P(Y \leq y) = P\left(\sigma_X\, Z \leq y\right) = P\left(Z \leq y/\sigma_X\right)$$
$$= \exp\left[-e^{-\alpha\left(\left(y/\sigma_X\right) - \left(u + \mu_X/\sigma_X\right)\right)}\right]$$
$$-\infty < y < \infty \tag{95}$$

where Equation 82 has been used. For the present example this cdf takes the specific form

$$P_Y(y) = \exp\left[-e^{-4.89((y/1.4 \times 10^7) - 9.69)}\right]$$
$$-\infty < y < \infty \tag{96}$$

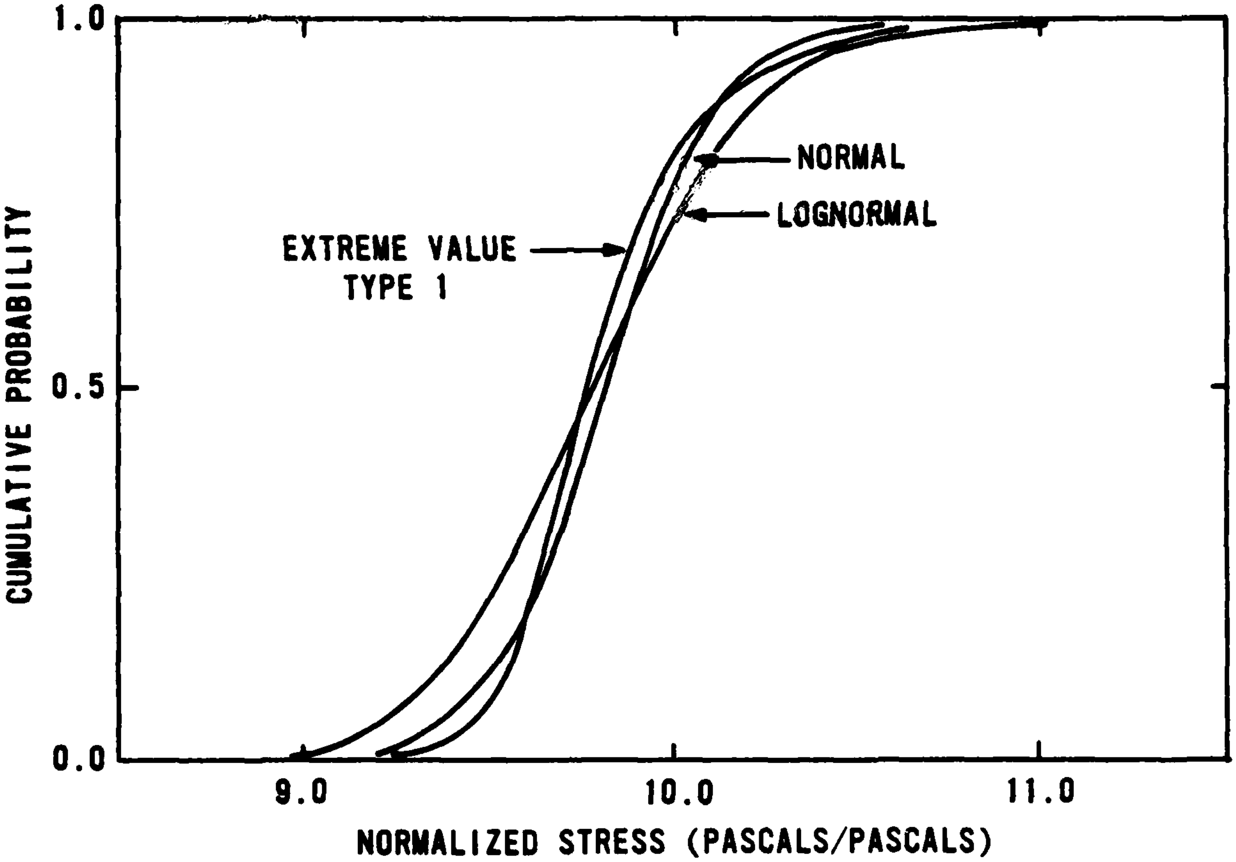

Figure 8. CDF's of the normalized highest peak load; extreme value Type I, normal, lognormal distributions.

Using this expression, the probability that the largest peak in the input, X(t), is lower than 15.0×10^7 Pa is approximately 0.993. This concludes the numerical example.

Note one important practical qualification on Equation 79, at this point. Equation 79 defines the average crossing rate of a zero mean, normal random process through the abscissa with positive slope. This quantity tells us about the frequency content of the random process, X(t), and indicates roughly how often the random process has an opportunity to assume a peak value (as opposed to a trough). When the probability distribution of the largest positive value assumed by a random process (as defined by Equation 76) is of interest, then the formulas presented above should be used. However, there is another important practical case. In some situations, it is necessary to find the probability distribution of the largest value in the random process, which is the absolute value of the random process considered above ($|X(t)|$), particularly when failure might be caused by either a positive or negative load. For example, when the load is a stress and the input random process, $X(t)$, has zero mean, and tension or compression can cause structural failure, then we are interested in the peaks of $|X(t)|$. In this case, define the peak normalized load as

$$Z = \max_{(o,T)} \frac{|X(t)|}{\sigma_X} \tag{97}$$

In this case,

$$E[N(0)] = 2 \cdot E[N_+(0)] \tag{98}$$

which is the expected number of zero crossing of the random process, $X(t)$, per unit time, should be used in Equation 80 in place of $E[N_+(0)]$. The only time it makes sense to use this approximation is when the mean of the underlying random process is zero; in this case, positive peaks and negative troughs are equally likely to cause failure. On the other hand, when $X(t)$ has a nonzero mean, it is most likely that failure causing loads will occur when the input load is increased beyond the mean. Therefore, when $X(t)$ has a nonzero mean, the crossing rate of Equation 79 should be used.

Once the probability distribution of peak load is known, it can be used along with the probability distribution of structural material strength in Equations 39 and 48 to find the structural reliability at a point. When the normal or lognormal probability distribution is used to charactrize structural material strength, the computation of Equations 39 and 48 must be done numerically. To avoid this necessity for numerical computations, some approximations can be made.

One possible approximation is simply to assume that the largest peak value probability distribution for $X(t)$ is a normal distribution. If we define Z, as in Equation 76, to be the peak normalized value of the random process, $X(t)$, above the normalized mean, realized in the time interval $(0,T)$, then the cdf of Z can be written

$$P\left(Z < z - \frac{\mu_X}{\sigma_X}\right) = \Phi\left(\frac{z - \left(\mu_Z + \mu_X/\sigma_X\right)}{\sigma_Z}\right), \quad -\infty < z < \infty \tag{99}$$

If the true largest peak value in the random process is defined using Equation 94, as before, then the cdf of Y is

$$P(Y < y) = \Phi\left(\frac{(y/\sigma_X) - (\mu_Z + \mu_X/\sigma_X)}{\sigma_Z}\right), \; -\infty < y < \infty \tag{100}$$

This is the normal cdf approximation to Equation 95. When the probability distribution of structural material strength is normal, then the reliability of a structure at a point can be determined using Equation 56.

For comparison, we repeat the numerical example presented above. An element is loaded so that its mean stress is $\mu_X = 7.0 \times 10^7$ Pa, and the spectral density of the oscillatory portion of the load is given in Figure 7. The load duration is 60 s; and Z is the random variable representing the greatest load on the element during the 60 s. (1) Find the mean and variance of the load on the structure, and (2) write an expression for the cdf of the peak load, Z.

By the assumption in the above paragraph, the mean and variance are the same in this case as in the previous case. These are

$$\mu_Z + \frac{\mu_X}{\sigma_X} = 9.81 \tag{101a}$$

$$\sigma_Z^2 = 0.069 \tag{101b}$$

The cdf of the normalized peak load is obtained by using these numbers in Equation 99.

$$P\left(Z < z - \frac{\mu_X}{\sigma_X}\right) = \Phi\left(\frac{z - 9.81}{0.263}\right), \; -\infty < z < \infty \tag{102}$$

This cdf is plotted in Figure 8 for comparison to the previous results. The probability that the actual largest peak in the input, Y, is lower than 15.0×10^7 Pa can be computed using Equation 100. It is approximately 0.9997. Apparently, the normal cdf approaches one more rapidly than the Type I extreme value cdf. This indicates that the results due to the normal assumption are less conservative than those connected with the extreme value distribution.

Another approximation for the distribution of the largest value in a normal random process is available. This is the lognormal approximation. Merchant, et al. (Ref. 3) have established a lognormal approximation for normal extremes which matches the Type I extreme value distribution at the fiftieth percentile and the 99.9 percentile points on the cdf's. This type of approximation is possible since the two parameters in the lognormal distribution can be chosen to force the type of fit described above. The approximation is executed as follows. First, define Z to be the normalized peak value realized by the normal random process, X(t), in (0,T). Assume that Z is lognormally distributed. Next, let $U = \ln Z$ be the normal random variable transformation of Z. (See Equations 61 through 63.) Compute the characteristic largest value of the underlying random process using Equations 79 and 80. The mean of the random variable U is given by

$$\mu_U = \ln\left[\Phi^{-1}\left(\exp\left[(1 - \Phi(u))(\ln 0.5)\right]\right)\right] \tag{103}$$

Where $\Phi(.)$ is the cdf of a standard normal random variable and $\Phi^{-1}(.)$ is its inverse. The standard deviation of the random variable, U, is given by

$$\sigma_U = 0.00199u - 0.0633 + 0.6634u^{-1} - 0.2648u^{-2} \tag{104}$$
$$1 \leq u \leq 16$$

These formulas come from Reference 3.

The probability that the normalized largest value exceeds the underlying normalized mean, μ_X/σ_X, by an amount less than z is

$$P\left(Z \leq z - \frac{\mu_X}{\sigma_X}\right) = \Phi\left(\frac{\ln\left(z - \mu_X/\sigma_X\right) - \mu_U}{\sigma_U}\right), \quad \frac{\mu_X}{\sigma_X} \leq z < \infty \tag{105}$$

When the true largest peak value in X(t), realized in the interval (0,T), is given by Y, as defined in Equation 94, then the cdf of Y can be written

$$P(Y \leq y) = \Phi\left(\frac{\ln\left(\left(y/\sigma_X\right) - \left(\mu_X/\sigma_X\right)\right) - \mu_U}{\sigma_U}\right), \quad \mu_X \leq y < \infty \tag{106}$$

These formulas establish a lognormal approximation to the probability distribution for the extreme values of a normal random process. When the probability distribution of structural material strength is lognormal, then the reliability of a structure at a point can be obtained using Equation 67.

For comparison we present the numerical example given twice, above. An element is loaded so that its mean stress is $\mu_X = 7.0 \times 10^7$ Pa, and the spectral density of the oscillatory portion of the load is given in Figure 7. The load duration is 60 s, and Z is the random peak load on the element. Then (1) find the parameters of the peak load used in the lognormal computations, and (2) write the cdf of normalized peak load.

The parameters of $U = \ln Z$ can be found by using $u = 4.69$ from Equation 86 in Equation 103. We obtain

$$\mu_U = 1.56 \tag{107a}$$

Equation 104 can be used to show that

$$\sigma_U = 0.075 \tag{107b}$$

In terms of these constants, the cdf of the normalized peak load is

$$P\left(Z < z - \frac{\mu_X}{\sigma_X}\right) = \Phi\left(\frac{\ell n(z - 5) - 1.56}{0.075}\right), \quad 5 < z < \infty$$

This cdf is plotted in Figure 8 for comparison to previous results. The probability that the actual largest peak in the input, Y, is lower than 15.0×10^7 Pa can be computed using Equation 106. It is approximately 0.9918. This result agrees quite well with the result obtained using the Type I extreme value distribution.

A summary of the material presented above is given in the final section of this chapter.

c. Distribution of the largest value in a sequence of random variables--The introduction to this section mentioned that applied structural loads might be either dynamic or static. Up to this point we have treated the dynamic case. The case where static loads from one random source are applied to a structure n times can be treated using the same equations. Let X_i, $i = 1, \ldots n$, be a collection of random variables representing random loads. Let the X_i be normally distributed with mean value μ_X and variance σ_X^2. Let a static load from each random source, X_i, be applied to a structure. Define the normalized peak random load as

$$Z = \max_i \frac{X_i - \mu_X}{\sigma_X} \tag{108}$$

The random variable, Z, is approximately governed by the cdf in Equation 77. That is, Z has, approximately, an extreme value Type I distribution. The characteristic largest value of Z is

$$u = \Phi^{-1}\left(1 - \frac{1}{n}\right) \tag{109}$$

Where $\Phi^{-1}(.)$ is the inverse function of the standard normal cdf. The parameters, α, μ_Z and σ_Z^2, can be evaluated using Equations 78 and 81. The actual value of the greatest load is governed by Equation 95. This distribution can be approximated by the normal (Eq. 100) and the lognormal (Eq. 106).

d. Peak distribution in a nonstationary random process--The results presented up to this point refer specifically to stationary normal random processes. It was mentioned at the beginning of this section that some techniques are available for finding the probability distribution of the largest peak load in a nonstationary random process. Two of these techniques will be discussed briefly.

All of the random loads which occur in nature are nonstationary, in a strict sense. This is true since no real load is in a steady state from the infinite past till the infinite future. However, we treat loads as though they were stationary when they assume a steady state character over a sufficiently long time duration. Exactly what duration is sufficiently long is left to the judgment of the analyst. For the present application, it is usually safe to assume that an input is stationary when its steady state duration is 100 times as great as the fundamental period of the structure excited by the input.

Under most circumstances it is conservative to assume that a random process structural load is stationary, when in fact it is not, if the spectral density of the structural load reflects the most severe portion of the load. In view of this, nonstationary random loads are often treated using the procedures outlined above. When this is done, two special conditions must be met. First, the spectral density of the load random process must be computed using the most energetic portion of the random process realization recorded. Second, the duration of the stationary load random process, used in computations, must be chosen to be longer than the severe portion of the measured input. This approach is often taken in the analysis of structural response to earthquakes, for example.

The second approach to treating nonstationary random load processes can be implemented only when several load records from a nonstationary random source are available. Assume that N records, denoted $x_j(t)$, $j = 1, \ldots N$, of a dynamic load are available, and are to be used to characterize the probabilistic nature of the highest peak in the random process which produced the $x_j(t)$. On each record we find the peak value, and we denote this X_j, $j = 1, \ldots N$. By definition

$$X_j = \max_t x_j(t), \; j = 1, \ldots, N \tag{110}$$

If we are interested in characterizing the peak values in the absolute value of the source, then the quantity, $|x_j(t)|$, should appear on the right hand side of Equation 110, in place of $x_j(t)$. The average of the highest peak in the underlying random load process can be estimated using the formula

$$\overline{X} = \frac{1}{N} \sum_{j=1}^{N} X_j \tag{111}$$

and the variance of the highest peak in the underlying random load process can be estimated using the formula

$$\hat{\sigma}_X{}^2 = \frac{1}{n-1} \sum_{j=1}^{N} (X_j - \overline{X})^2 \tag{112}$$

At least ten records should be used in performing these estimates, otherwise serious inaccuracies can occur. The probabilistic behavior of the highest peak can be determined by assuming that the highest peak value obeys an extreme value Type I distribution, or a normal distribution or a lognormal distribution. The moments estimated in Equations 111 and 112 must be used to find the parameters that are used in the cdf of whichever model is chosen.

3. RELIABILITY CONCEPTS

Previous sections of this chapter discussed fundamental aspects of probability theory and random process theory which relate to solution of the elementary reliability problem at a point. It has been shown that, when the probability distributions of structural material strength at a point and load at the same point are known, then Equations 39 and 48 can be used to compute the structural reliability at that point. When a structure is to be exposed to a field environment which can cause that structure to fail at one point, then the approaches developed in the previous sections are sufficient to compute the structural reliability. On the other hand, when the field loads are sufficient to create the potential for failure at two or more points, then the approaches developed previously are generally not sufficient to compute the probability of structural survival.

In view of this, two questions must be answered. First, how many points must be considered in a given structural reliability analysis? Second, how is structural reliability computed when a potential for failure exists at many points? In the following, these questions are answered in sequence.

a. Structural locations to be considered in reliability analysis--When a structure is loaded mechanically, a material failure can occur in any of the senses discussed in paragraph I-1. This possibility exists because the probability of failure is not zero when an element is loaded. This fact can be seen from the reliability formulas and the numerical examples presented in the previous sections. The probability of failure is usually extremely small at all but a very few points, however. There is a method which can be used to determine which points should be included in a reliability analysis. Before outlining this method, though, an important practical point should be made.

Within a particular simple structural component it is not usually necessary to consider the possibility of failure at more than one point, unless the random material variation is rapid, in a spatial sense, on the component. We demonstrate this point with an example. Consider a simply supported beam in Figure 9. The beam is loaded at its center with the deterministic load, F.

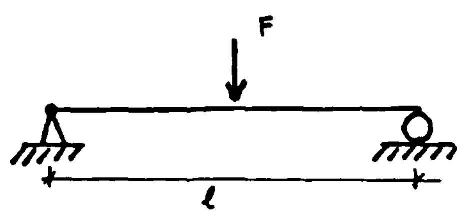

Figure 9. Simple beam with concentrated load applied at center.

The section depth is 2c and the cross-section moment of inertia is I. The outer fiber stress under the load is

$$\sigma\left(\frac{\ell}{2}\right) = \frac{F\ell c}{4I} \tag{113}$$

If we denote the material failure strength random variable at the outer fiber at the beam center as Y, then the probability of failure is

$$P\left(Y < \sigma\left(\frac{\ell}{2}\right)\right) = P_Y\left(\frac{F\ell c}{4I}\right) \tag{114}$$

At any other point on the beam, the outer fiber stress is

$$\sigma(x) = \begin{cases} (Fc/2I)x\,, & 0 < x < \ell/2 \\ (Fc/2I)(\ell - x), & \ell/2 < x < \ell \end{cases} \tag{115}$$

where x is a coordinate starting at the left end of the beam. If the material failure strength random variable is the same at all other points as it is at the center, then the probability of failure at point x is

$$P(Y < \sigma(x)) = P_Y(\sigma(x)),\ 0 < x < \ell \tag{116}$$

But since $\sigma(x) < \sigma(\ell/2)$ except when $x = \ell/2$, the probability of failure is lower when $x \neq \ell/2$. Therefore, in this case it is reasonable to consider the possibility of failure at only one point in a simple structural component. Generalize the result to state that whenever the failure strength is the same at all points on a simple component, it is necessary to consider the possibility of failure only at the location most severely stressed.

We note, further, that it is reasonable to assume that anv simple component made from material which comes from one source, produced at one time, may be considered to have a single material failure level. This is only an approximation since real material is not perfectly homogeneous; however, it should be reasonably accurate, and it considerably simplifies reliability analyses. Components fabricated from material obtained from different sources should be considered to have different failure levels.

A point which was developed in paragraph II-2, and which was made most clear in the numerical examples demonstrating the normal distribution, is that the reliability at a point on a structure is indexed by the quantity

$$q = \frac{\mu_Y - \mu_X}{\sqrt{\sigma_X^2 + \sigma_Y^2}} \tag{117}$$

where μ_Y and μ_X are the mean values and σ_Y^2 and σ_X^2 are the variances of the random strength and load at a point on a structure. (Note that the random variable, X, and its moments, μ_X and σ_X^2, refer to the highest peak load applied to a structure during its design life.) In fact, when the cdf of a standard normal random variable is evaluated at q, the result is the reliability of a structure at a point. Reference to Table 1 in Reference 1 shows that as q increases, the reliability at a point increases rapidly. This demonstrates our contention that q is an index of reliability at a point. Note that q is a direct measure of structural reliability only in the case where load and strength are normally distributed, but in all cases q can be used as an index of reliability.

Table 1 shows the values of reliability which correspond to specific values of q when the load and strength are normally distributed. When q goes to zero at a point on a structure, the probability of failure approaches the value 0.5. Failure at a point is clearly most likely to occur at those points on components where the reliability index, q, is lowest.

TABLE 1. RELIABILITIES OF STRUCTURES WITH VARIOUS INDEX VALUES

q	R
0	0.5000
1	0.8413
2	0.97724
3	0.99865
4	0.99997
5	0.9999997

R is the reliability of a single component structure with normal load and strength and reliability index, q.

Under most circumstances when the reliability of a structure is computed, the result is dominated by the influence of point reliability at points on the structure where the reliability is lowest. In view of this, in the reliability analysis of a complex structure it is most important to consider the potential for failure at those places where the reliability index, q, is lowest. In order to determine which points must be considered in a reliability analysis, use the following procedure.

a. Find the point on each component of a structure which is critical for a loading configuration.

b. At each of these points find the reliability index, q.

c. Note the smallest value of q; this is q_0.

d. Refer to Table 2; enter the table with the approximate number of components in the system and the value of q_0. Read off a value of q_1.

e. All components with reliability indices, q, in the range (q_0, q_1) must be considered in a reliability analysis.

TABLE 2. RELIABILITY INDEX INTERVALS

	Number of Components		
q_0	10	100	1000
1.0	2.95	3.60	4.16
1.2	3.05	3.69	4.23
1.4	3.15	3.78	4.32
1.6	3.27	3.78	4.32
1.8	3.39	3.98	4.49
2.0	3.51	4.08	4.59
2.2	3.64	4.19	4.69
2.4	3.77	4.31	4.80
2.6	3.92	4.43	4.91
2.8	4.07	4.56	5.02
3.0	4.22	4.70	
3.2	4.37	4.83	
3.4	4.52	4.99	
3.6	4.67		
3.8	4.82		
4.0	4.98		

Values of q_1 are listed in the table.

For values not covered in this table, and when $q_0 > 3$, use the following formulas:

$$q_1 > \Phi^{-1}\left\{\left[1 - Q(1 - \Phi(q_0))\right]^{1/n}\right\}$$

where Q = 0.10 has been used to derive the values in the table, above, and

$$\Phi(X) \cong \frac{1}{2} + \frac{1}{2}\left(1 - e^{-2X^2/n}\right)^{1/2}$$

$$\Phi^{-1}(p) \cong \sqrt{-\frac{\pi}{2}\,\ell n\left[1 - (2p - 1)^2\right]}$$

The foregoing approach to determination of how many elements must be included in a reliability analysis is based on the following reasoning. Let a structure be composed of n independent elements. Let the reliability of the least reliable element be R_0, and let the reliability of each remaining element be equal to or greater than R_1. The reliability index of the least reliable element is $q_0 = \Phi^{-1}(R_0)$, and the reliability index of each remaining element is $q_1 > \Phi^{-1}(R_1)$. The probability of failure of the least reliable element is

$$p_0 = 1 - R_0 = 1 - \Phi(q_0) \tag{118}$$

and the probability of failure due to all remaining elements is

$$p_1 < 1 - R_1^{n-1} = 1 - [\Phi(q_1)]^{n-1} \tag{119}$$

if we require that

$$p_1 < Qp_0 \tag{120}$$

and we set Q equal to some small number, then the probability of failure relating to the n - 1 more reliable elements is a factor of Q smaller than the probability of failure due to the least reliable element. In deriving Table 2 we have set Q = 0.1, and used the expressions of Equations 118 and 119 in Equation 120. Then we solved for q_1 and listed this in the table.

This derivation is limited if the element probabilities of failure are not independent in a complex structure. However, we believe that the results are conservative in providing a guideline regarding the number of elements that should be included in an analysis.

When only the one element with reliability index q_0 falls in the interval (q_0, q_1), then only this element need be considered in a reliability analysis. From a design viewpoint, a well balanced design will have many elements with reliability indices in the interval (q_0, q_1).

b. Three approaches to reliability analysis--Once the collection of elements to be considered in the reliability analysis is determined, the analysis can be executed. The probabilistic relations among the loads to be applied to the structural elements and the strength of the elements is very important in reliability analysis. With reference to these quantities, a reliability analysis can follow numerous lines, three of which are described here. (These cases were chosen because they can all be treated using the computer program FSR.)

(1) One can assume that all structural loads are mutually independent of all strengths, all loads are mutually independent of one another, and all strengths are mutually independent of one another. Analyses performed using this assumption are simple and can be shown to yield conservative results.

(2) One can assume that all loads are mutually independent of all strengths, all strengths are mutually independent of one another, and all loads have some finite constant degree of correlation with one another.

(3) One can assume that all loads and strengths are mutually independent, but the loads have arbitrary levels of correlation with one another, and the strengths have arbitrary levels of correlation with one another.

The following paragraphs discuss computation of reliability using each of the assumptions listed above. The following section describes the input which must be obtained for use in FSR when an analysis using each assumption is executed.

As mentioned previously, it is not usually necessary to consider the structural reliability at two points on a simple structural component, but only at the point which is most critically loaded. The reason for this is that the strength at one point on a simple component is a good indicator of the strength at an adjacent point. Further, it is reasonable to assume that there is no correlation between the failure strength of separate components. Therefore, as far as element strength is concerned, it is reasonable to assume independence between components. The same statement cannot usually be made for loads, though; the reason for this will be demonstrated in a later paragraph. In spite of this, mutual independence between pairs of loads in a structural reliability analysis is often assumed. There are at least two reasons for this. One reason is that this assumption considerably simplifies a reliability analysis, as will be shown. A second reason is that the results obtained using this assumption are conservative. That is, the reliability predicted using this assumption is lower than the true reliability of a structure.

The reliability of a structure is the joint probability that each of the points considered in the analysis will survive the load applied to it. This implies that if failure occurs at one point, then the entire structure fails.

This is known as a weakest link failure assumption, and is conservative for statically indeterminate structures. This definition of structural reliability is used throughout this report. Since independence has been assumed between pairs of structural components, this joint probability can be written as a product of marginal probabilities.

The reliability analysis can be executed as follows.

a. The mean and variance of load and structural material strength at the critical point on each structural component are computed.

b. The reliability index, q, at each point is computed and the collection of points to be included in the reliability analysis is determined, as described above. Let the points to be considered in the analysis be indexed, $j = 1,\ldots N$.

c. The reliability at each point is computed using one of the formulas from paragraph II-1 (Eqs. 39 or 48). Let the reliability at point j be denoted, R_j, $j = 1, \ldots N$.

d. The overall structural reliability, R, is computed using the formula

$$R = \prod_{j=1}^{N} R_j \tag{121}$$

The probability of failure of the overall structure is simply the complement of the structural reliability; this is

$$p_f = 1 - R \tag{122}$$

The simple multiplication of point reliabilities, used in Equation 121 is made possible by the assumption of independence among all loads and strengths. Given the means and variances of loads and structural material strengths at the points to be considered in the analysis, the FSR program will execute the computations given above for the particular case where the loads and strengths are normally distributed. Details of the input are described in the next section.

Note again that the reliability analysis performed using the total independence assumption provides a conservative result; that is, the estimated reliability forms a lower bound on the true structural reliability.

When assumptions are made that the structural loads and strengths are mutually independent, and the strengths at various structural points are mutually independent, and the input loads are correlated where every pair of loads has the same correlation coefficient, the problem of estimating the system reliability becomes considerably more complicated than the problem solved above. Specifically, the simple multiplication rule used in Equation 121 is no longer valid. Rather, a more complicated multidimensional integral of a multidimensional pdf is required. The dimension of the pdf is equal to the number of components included in the reliability analysis.

The problem to be solved here can be expressed in the following way. Let Z_j, j = 1, ... n, be the difference between the structural material strength and the peak load applied at point j, and n the number of points included in the analysis. The probability of survival is the reliability of the structure, and this is the chance that each Z_j, j = 1 ... n, is greater than or equal to zero. This is written

$$R = P(Z_1 > 0, Z_2 > 0, \ldots, Z_n > 0)$$

$$= \int_0^\infty dz_1 \int_0^\infty dz_2 \cdots \int_0^\infty dz_n \, p_{Z_1, Z_2 \ldots Z_n} (z_1, z_2, \ldots, z_n) \qquad (123)$$

In the important case where the pdf is normal, this integral cannot be evaluated in closed form. Therefore, a numerical approach must be taken to the solution of this problem. Merchant, et al., have shown in Reference 3 that the integral expressed above can be evaluated numerically using an approach which converts the n fold integral into a collection of sequences of univariate integrals, where each sequence has n integrals. This conversion can be performed when each pair of load random variables has a correlation coefficient with the same magnitude. The technique is implemented in the FSR computer program. This conversion is practically useful since it allows us to analyze systems with hundreds of elements. Reference to Equation 51 indicates how complicated a direct solution of Equation 123 would be. When the system correlations are not all equal, a direct numerical analysis of Equation 123 can only be used to analyze systems of size up to about n = 4.

The method of analysis used to evaluate Equation 123 depends on the assumption that every pair of load random variables is correlated with the same correlation coefficient. While input loads may be correlated, it is not usually the case that the correlation coefficient is invariant from one pair of loads to the next. Correlation between load random variables arises from many sources and two of these are discussed in the following, and a method for computing input load correlation is presented.

First, when a mechanical structure is loaded statically and the loads assume a given physical configuration, and the magnitudes of the loads vary so that the ratio between any pair of loads is a constant, then the stresses at all points in the structure vary in such a way that the ratio between any two stresses is a constant. If the magnitude of the input loads is a random variable, then the correlation between each pair of loads is perfect, and the correlation coefficient between each pair of loads is +1 or -1. Correlation is positive if both loads are tensile at one time or compressive at one time. Correlation is negative if one load is tensile while the other is compressive.

When static loads are applied to a structure in a given physical configuration and the load magnitudes are random, and the ratios of pairs of loads is also random, then the correlation between pairs of loads will not be 1. The correlation between a pair of loads depends on how nearly the ratio of the magnitudes of the pair of loads remains constant. If the ratio of magnitudes of a pair of loads is always nearly constant, then the correlation coefficient between the loads is near ±1. If the ratio of magnitudes of a pair of loads is highly variable, then the correlation coefficient between loads is near zero.

The correlation coefficient between a pair of static loads, F_1 and F_2, can be estimated as follows when a collection of data measured in the field is available. Let F_{1j} and F_{2j}, j = 1, ... n, be the stresses measured at locations 1 and 2, respectively, during n random experiments. The mean values of F_1 and F_2 can be estimated using the formula

$$\overline{F}_i = \frac{1}{n}\sum_{j=1}^{n} F_{ij}, \quad i = 1,2 \tag{124}$$

The variances of F_1 and F_2 are estimated using the formula

$$\delta_{F_i}^{2} = \frac{1}{n-1}\sum_{j=1}^{n}\left(F_{ij} - \overline{F}_i\right)^2, \; i = 1,2 \tag{125}$$

The estimated standard deviations of the loads F_1 and F_2 are simply the square roots of the estimated variances. The covariance between the loads F_1 and F_2 is estimated using the formula

$$\hat{K}_{12} = \frac{1}{n} \sum_{j=1}^{n} \left(F_{1j} - \overline{F}_1\right)\left(F_{2j} - \overline{F}_2\right) \tag{126}$$

Given the above estimates, the correlation coefficient between the loads, F_1 and F_2, is estimated

$$\hat{\rho}_{12} = \frac{\hat{K}_{12}}{\hat{\sigma}_{F_1} \hat{\sigma}_{F_2}} \tag{127}$$

As mentioned in paragraph II-1, this is a number between +1 and -1. It is clear from Equation 126 that, if F_1 and F_2 both tend to be on the same side of their means, then $|\hat{\rho}_{12}|$ has a value near 1. When F_1 and F_2 bear no linear relation to each other, then $\hat{\rho}_{12}$ is near zero.

A second situation where structural loads are correlated with one another occurs when a structural system is excited by random dynamic loads. When the input loads are stationary, the correlation between the stresses $F_1(t)$ and $F_2(t)$, at two points, can be estimated using the general procedure outlined above. In the present case, though, separate stress records will not be available for separate applications of static loads. Rather, sample records of the random processes, $F_1(t)$ and $F_2(t)$, will be available. These sample records must be digitized using a time interval, Δt. Let $t_j = j\Delta t$, $j = 1, \ldots n$. Identify the stress at time t_j, $F_i(t_j)$, as F_{ij}, $i = 1,2$. Then use Equations 124 and 125 to estimate the load stress means and variances, respectively. Equation 126 is used to estimate the covariance between the stationary random processes, $F_1(t)$ and $F_2(t)$. Finally, Equation 127 is used to estimate the correlation coefficient between the input random processes. As in the static case, the correlation coefficient between the input stress random processes, $F_1(t)$ and $F_2(t)$, is a number between -1 and +1. The implications of the correlation coefficient values are the same as before.

Correlation between structural loads can arise from a third source. Often in a reliability analysis when a single structural component is loaded in multiple modes, one assumes that the single component acts as multiple components, one for each mode of loading. For example, a component may be loaded in two directions; then one can assume that the component acts as two components, one for each direction of loading. Or a component may be loaded in shear and bending, and one can assume that it acts as two components, one to resist shear and one to resist bending. When a single external load results in two internal structural member loads which are linear functions of the external load, then the internal member loads are perfectly correlated ($\rho = \pm 1$). When the internal member loads are nonlinear functions of the external load, then the internal load correlation coefficient is not 1.

Note that there are theoretical approaches for computing the correlation coefficient between pairs of random variables and pairs of random processes when the forcing input random variables (static case) or forcing input random processes (dynamic case) are completely characterized, and when the structural parameters are known. These approaches are rather involved, mathematically, and therefore are not discussed in this report. Consult References 6 and 7 for discussions of these analytical techniques.

As stated previously, the load to be considered in a reliability analysis is the peak load applied to a structure. Whether multiple static loads are applied to a structure or a dynamic load is applied to a structure, the load pdf used in the reliability equation reflects the peak load applied to a structure. In view of this, the correlation coefficient used in a reliability analysis (when independence is not assumed) must be the correlation coefficient between peak loads at two points. In Reference 3, Merchant, et al., have shown how to obtain the correlation coefficient between pairs of peak loads. The correlation coefficient, ρ_E, between the largest values assumed by a pair of random variables, F_1 and F_2, or random processes, $F_1(t)$ and $F_2(t)$, can be estimated using the formula

$$\rho_E = \rho_N(\alpha + \beta u^{\gamma}) \tag{128a}$$

where ρ_N is the correlation coefficient between F_1 and F_2 or $F_1(t)$ and $F_2(t)$. α, β and γ are parameters with values

6. Lin, Y. K., Probabilistic Theory of Structural Dynamics, McGraw-Hill Book Company, New York, 1967.

7. Newland, D. E., An Introduction to Random Vibrations and Spectral Analysis, Longman Group Limited, London, 1975.

$$\alpha = 0.305\, \rho_N^4 + 1.181 \tag{128b}$$

$$\beta = 0.906\, \rho_N - 0.0476 \tag{128c}$$

$$\gamma = 1.023\, \rho_N^4 + 0.8352 \tag{128d}$$

The quantity, u, is the normalized characteristic largest value of the random variables or random processes. In the case where multiple static loads are applied to the structure, u can be obtained using Equation 109. When dynamic loads are applied, u can be obtained using Equation 80. In the latter case, the u values may differ for the random processes $F_1(t)$ and $F_2(t)$; in this case the average should be used. Equation 128 relies on the assumption that the inputs are normally distributed.

In most practical situations it will happen that the correlation coefficients between different pairs of loads have different values. Loads that are applied at locations close to one another may tend to be strongly correlated. Other pairs of loads may be weakly correlated or uncorrelated. For any group of structural elements, the reliability analysis method under consideration allows only one value of correlation coefficient to be taken into account. In view of this, the reliability computation must be performed as follows. The structural elements, being considered in an analysis, must be divided into groups. Each group should contain, as far as possible, elements with highly correlated loads. And the loads on one group of elements should have a relatively low correlation with the loads on every other group of elements. Then the reliability of each group of elements should be computed using Equation 123, for example, in the FSR program. The single value of correlation coefficient between pairs of loads should be chosen as the average of the correlation coefficient between all pairs of loads in the group. If there are N groups of components, and the reliability of the components in the j^{th} group is R_j, then the overall structural reliability can be computed from

$$R = \prod_{j=1}^{N} R_j \tag{129}$$

This formula is possible since the groups of loads have been chosen so that individual loads in different groups have a low correlation.

The probability of failure of the overall structure is

$$p_f = 1 - R \tag{130}$$

When the means and variances of the loads and structural material strengths to be considered in an analysis are known, and when the correlation coefficients between pairs of loads at analyzed points are known, then the type of analysis discussed above can be executed using FSR. This program assumes all loads and strengths to be normally distributed. Details required for preparing input for use in FSR are given in Section III.

Note again that the most conservative analysis is that in which all loads and strengths are assumed independent; therefore, the results obtained in Equation 121 should always provide a lower bound on the results obtained using Equation 129, for a given system. For this reason, the results obtained by the analysis ending at Equation 121 may be considered preferable in view of the additional effort required to obtain the results given above.

The most general case that could possibly be considered in a reliability analysis is one in which all loads and structural material strengths at the analyzed points are arbitrarily correlated. While a capacity to perform this type of analysis may be desirable, it is almost inconceivable that it could be put to use in the context of the problems considered in this report. The main reason for this is that structural loads and strengths cannot normally be correlated. This is easy to see when one considers that, in order for loads and strengths to be correlated, above average loads would have to be applied either to components with consistently above average strength or to components with consistently below average strength. It would be practically impossible to arrange for this.

The reliability analysis considered at present is that in which the structural peak loads, applied to the components analyzed, have arbitrary correlation, the components have strengths with arbitrary correlation, and the loads and strengths are uncorrelated. In this case it is required to integrate Equation 123. Given the requirements on correlation, the integral cannot be evaluated in closed form when the loads and material strength are normally distributed. The integral can be evaluated numerically when the loads and strengths are normally distributed, but due to practical difficulties, discussed in Reference 3, the number of elements considered in the reliability analysis is limited to about four. This is a small number of elements, but may be sufficient for some practical analyses.

The capability to perform the reliability analysis described above adds two features to the reliability analysis discussed previously. First, it allows the use of arbitrarily correlated load random variables. Second, it allows correlation between structural material strength random variables. One technique for the estimation of the correlation coefficient between random loads is discussed above. The results of this type of estimation or any other can be used to obtain input for analysis, for example, in the FSR computer program.

However the correlation coefficients between strength random variables are obtained, it will be found that the correlation coefficients between strengths of components fabricated from materials obtained from different sources will be zero. In general, only those correlation coefficients which consider material strengths at two different points on one component, fabricated from one material sample, will be nonzero. If the material in the component under consideration is of a high quality and very uniform, than the correlation coefficient should be very near 1. Lower quality material samples may be less uniform and the strength correlation coefficients between different points may be lower. For a particular material sample the strength correlation coefficient should always be positive.

As stated previously, generally only one point on a simple component need be considered in a reliability analysis, and this point is the critically loaded point. There are some situations, however, where this is not true. Specifically, in a reliability analysis one should consider multiple points on any component which has two or more points which are severely loaded and whose reliability indices, q, are close in value.

One method for finding the correlation coefficient between material strengths, F_1 and F_2, at different points on a simple component is to test many sample components to the point of failure. The results of the tests can be used to estimate the strength correlation coefficient. Assume that n tests are performed and that failure stresses are recorded at two points during each test. These are denoted F_{1j} and F_{2j}, j = 1, .. n. The means and variances of the failure loads, $\overline{F}_1$ and $\overline{F}_2$, and $\hat{\sigma}_{F_1}{}^2$, and $\hat{\sigma}_{F_2}{}^2$, can be estimated using the Equations 124 and 125, respectively. The covariance between failure loads, K_{12}, can be obtained using Equation 126. Finally, the correlation coefficient between failure loads can be estimated using Equation 127. This estimate is obtained for all correlated pairs of failure strengths, and used in the pdf of Equation 123. The equation is numerically integrated, thereby yielding the structural reliability. The details of integration of Equation 123 in the case where the loads and strengths are normally distributed and arbitrarily correlated are discussed in Reference 3; because of their complicated nature they will not be discussed here. The type of numerical analysis discussed above is implemented in the FSR computer program for the normal load and strength case.

Note that the correlation between material strengths at different points on a structural component can be obtained using a theoretical approach when a model for probabilistic material strength and the structural geometry are given.

The fact that the reliability analysis discussed above is limited in the number of elements which can be considered, may make one of the reliability analysis approaches discussed previously more desirable. When independence among all loads and strengths is assumed, then the results of the analysis are conservative.

4. SUMMARY OF FORMULAS

In paragraphs 1, 2, and 3 of this section, formulas for computing the reliability of a point on a structure, and for computing the overall reliability of a structure, were presented. Some special formulas were also given for computation of reliability when the load and strength random variables are normally distributed or lognormally distributed. An outline for the sequential use of these formulas is presented in this section.

The first step in a structural reliability analysis is to determine the moments of the largest loads applied at critical structural locations, and to determine the moments of the strength at those points. This information can be obtained from data collection and direct statistical analysis, or from a theoretical analysis. In the process of determining load moments we assume a distribution for the largest peak load. The moment computation and load cdf specification proceed as follows. When a stationary normal random process is applied to the structure under consideration, the probability distribution of the largest peak value of the input can be obtained using the formulas of paragraph II-2, when the spectral density and mean value of the load are available at the point to be analyzed. Specifically, Equations 79 and 81 can be used to find the parameters of the Type I extreme value distribution which characterizes the largest peak load. The cdf of this random variable is given in Equation 95. The mean and variance of the largest peak load can be estimated using Equation 78. These parameters can be used to characterize a normal distribution of the largest peak load, if this assumption is chosen rather than the extreme value Type I distribution. The normal distribution of the largest peak load is given in Equation 100. Finally, if the lognormal approximation of the distribution of the largest peak load is desired, then Equations 103 and 104 can be used to find the parameters. The lognormal cdf of the largest peak load is given in Equation 106.

When the means and variances of the peak loads and material strengths at critical structural locations are known, the reliability index can be established at every point on the structure where a failure potential exists using Equation 117. The lowest valued reliability index is noted and this is used, along with the number of elements in the structure and Table 2, to determine the range of the reliability index values for elements which must be included in the reliability analysis.

At this point, one of the assumptions listed in paragraph II-3-b regarding load and component independence must be made. For preliminary purposes, it is probably easiest to assume independence between all load pairs and all pairs of structural material strengths. When this is done, the reliability at each point must be found. This can be done by evaluating the reliability integral in Equation 39 or 40. When the load and strength are assumed normally distributed, the expression of Equation 57 can be used to compute the structural reliability. When both the load and strength are assumed lognormally distributed, then the expression of Equation 67 can be used to estimate the structural reliability. When the reliability of each component is known, then Equation 121 can be used to find the overall structural reliability. This estimate is a lower bound on the true reliability of the structure. If the estimate shows that the reliability is high enough, then no further computations need be performed. Some of the reliability computations mentioned above can be performed using the FSR computer program. Specific inputs for this program are listed in the following section.

When the reliability estimate obtained above is not satisfactorily high, then a more accurate approximation of the structural reliability can be sought.

This might be obtained by considering the correlation between loads on the structure. Equations 124 and 127 can be used to estimate the correlation coefficient between structural load pairs. With this information a reliability estimate can be obtained through integration of Equation 123. As explained in the text preceding Equation 129, groups of elements with highly correlated loads must be established. The reliability of these groups can be evaluated using a special technique from Reference 3, in the special case where the loads and strengths are represented as normal random variables. The overall structural reliability is found by taking the product of the reliabilities of the individual groups, as in Equation 129. This computation is implemented in the FSR computer program.

The most accurate reliability analysis which can be performed takes into account the correlations between structural loads and the correlations between structural material strengths. When the number of elements to be considered in an analysis is low, say about four, then this accurate approach can be taken. The correlation coefficients between structural material strengths can be estimated using Equations 124 through 129. The correlation coefficients between loads are obtained as described in the previous paragraph. With this information, the pdf of Equation 123 can be established and this can be numerically integrated. The procedure for doing this is outlined in Reference 3 for the case where the load and strength random variables are normally distributed. This numerical procedure is implemented in the FSR computer program.

A practical feature of the FSR computer program is that the reliability can be computed for a complex structure in which the components can be divided into three types of groups. The first group type contains elements, with uncorrelated strengths, that are loaded with uncorrelated loads. The second group type contains elements, with uncorrelated strengths, that are loaded with arbitrarily correlated loads. The third type has elements, with arbitrarily correlated strengths, that are loaded with arbitrarily correlated loads. In all groups the loads and strengths are uncorrelated.

In establishing the capability to analyze structures whose loads and strengths are normally distributed, we also establish the capability to analyze structures whose loads and strengths are lognormally distributed. To use the FSR computer program to analyze problems where the loads and strengths are lognormally distributed, we simply input the mean values and variances of the logarithms of the load and strength random variables. The transformation of the mean and variance of a random variable into the mean and variance of the logarithm of the random variable can be accomplished using Equation 62. When correlation coefficients are used in a lognormal reliability analysis, the correlation coefficients must be entered between the logarithms of the random variables under consideration, in order to be strictly correct. It can be shown, though, that the correlation coefficient between the logarithms of two random variables is well approximated by the correlation coefficient between the two random variables themselves. An improved approximation is given in Reference 3.

III. INPUT FOR THE FSR COMPUTER PROGRAM

This section describes the inputs that must be used to execute the two types of analyses done by the computer program, FSR. The explanations provided here are summaries of the material presented in Appendix A of Reference 3.

1. RELIABILITY ANALYSIS OF COMPLEX STRUCTURES

First, outline one form of the input that can be used to perform a reliability analysis of a complex structure. The analyst wishing to use FSR should consult Appendix A in Reference 3 for alternate forms of input. Table 3 lists the card inputs for FSR. An explanation for every input in the table is given immediately following the table. When more than one card may be needed to enter the data, this is noted. Numerical data inputs may be separated by commas or spaces. Where required, a note is included to indicate that the input should be an integer (numerical constant entered without a decimal point). NTOTAL is the total number of components considered in the analysis of the structure. NTOTAL must be a number lower than 2000.

Some numerical examples demonstrating the use of this part of the FSR computer program with the inputs, described in Table 3, are given in Section IV.

2. COMPONENT FACTOR OF SAFETY AND RELIABILITY

The computer program FSR can be used to execute a second type of analysis. This is a computation of the factor of safety of a single structural component. Option 1 of the program computes the factor of safety for the case where the load and strength probability distributions are both normal or both lognormal. This involves evaluation of an expression provided in Reference 3. Option 2 of the program computes the factor of safety for the case where the load and strength probability distributions are chosen from a list including the normal, lognormal and extreme value Type 1 distributions, and others listed in Appendix A, Reference 3. The load and strength probability distributions need not be the same.

Describe the input for Option 1 of the program, first. The cards needed to run Option 1 are listed in Table 4. An explanation of each card is given immediately following the table. The format that must be used for the input is listed with each card.

TABLE 3. INPUT DATA CARDS FOR STRUCTURAL RELIABILITY ANALYSIS USING FSR

	Column 1
Card 1	FSR3
Card 2	1, NUNC, NCORR, MAXM, NEQM (These are integers)
Card 3	NRHO(1), NRHO(2), ... NRHO(NEQM) (These are integers. Multiple cards may be needed. If NEQM = 0, enter zero (0).)
Card 4	RHOL(1), RHOL(2), ... RHOL(NEQM) (Multiple cards may be needed. An additional card (or set of cards) may be required following card 4. The requirement is explained below under "RHOL(I)." If NEQM = 0, enter a zero (0).)
Card 5	A(1,1), A(1,2), ... A(1, NCORR), A(2,2), ... A(2, NCORR), ... A(NCORR, NCORR) (Multiple cards may be needed. If NCORR = 0, skip cards 5 and 6).
Card 6	B(1,1), B(1,2) ... B(1, NCORR), B(2,2), ... B(2, NCORR), ... B(NCORR, NCORR) (Multiple cards may be needed).
Card 7	SVS(NCORR + 1), SVS(NCORR + 2) ... SVS(NTOTAL) (Multiple cards may be needed if NCORR = NTOTAL, skip cards 7 and 8.)
Card 8	SVL(NCORR + 1), SVL(NCORR + 2), ... SVL(NTOTAL) (Multiple cards may be needed.)
Card 9	MUS(1), MUS(2), ... MUS(NTOTAL) (Multiple cards may be needed.)
Card 10	MUL(1), MUL(2), ... MUL(NTOTAL) (Multiple cards may be needed.)

The input terms are defined as follows:

NUNC The number of components whose strengths are uncorrelted with the strengths of other components and whose loads are uncorrelated.

NCORR The number of components whose strengths are arbitrarily correlated and whose loads are arbitrarily correlated. (This should be limited to four.)

MAXM This parameter limits the number of terms which can be included in a series solution of Equation 123; it should be shown as 4.

TABLE 3. (Continued)

NEQM The number of groups of components whose strengths are uncorrelated and whose loads are equally correlated.

NRHO(I) The number of components in the I^{th} group of components whose strengths are correlated and whose loads are equally correlated.

RHOL(I) The correlation coefficient of the I^{th} group of components whose strengths are uncorrelated and whose loads are equally correlated. For each number on card 4 which is negative, an additional card (or cards) must be included following card 4. This additional card indicates the polarity of the load random variables in a particular group by a +1.0 or -1.0. On each of the additional cards a sequence of +1.0's and -1.0's is listed showing the sign of the correlation coefficient between the first component in the group and each other component in the group. For example, let a group be composed of four components and let the correlation coefficients between the load on the first component and the loads on the other components be ρ_{12} = -0.5, ρ_{13} = -0.5, ρ_{14} = 0.5. Let the group be numbered 8. Then RHOL(8) = -0.5, and a card corresponding to this correlation coefficient would be included following card 4, and on this card the following numbers would appear.

+1.0 -1.0 -1.0 +1.0

(The first +1.0 shows the sign of the correlation of the load on the first component with itself; this is always +1.0.)

A(I,J) The element in the I^{th} row and J^{th} column in the loads covariance matrix for the structural components with arbitrarily correlated loads and strengths. The entire matrix is shown below. Only the upper triangle of the matrix is read in.

$$\begin{bmatrix} A(1,1) & A(1,2) & \cdots & A(1,NCORR) \\ A(2,1) & A(2,2) & \cdots & A(2,NCORR) \\ \vdots & & & \vdots \\ A(NCORR,1) & A(NCORR,2) & \cdots & A(NCORR,NCORR) \end{bmatrix}$$

The first row, from A(1,1) to A(1,NCORR), is read in first. The second row, from A(2,2) to A(2,NCORR), is read in next, etc.

TABLE 3. (Continued)

B(I,J) The element in the I^{th} row and J^{th} column in the strength covariance matrix for the structural components with arbitrarily correlated loads and strengths. The entire matrix is shown below. Only the upper triangle of the matrix is read in.

$$\begin{bmatrix} B(1,1) & B(1,2) & \cdots\cdots & B(1,NCORR) \\ B(2,1) & B(2,2) & \cdots\cdots & B(2,NCORR) \\ \vdots & & & \vdots \\ B(NCORR,1) & B(NCORR,2) & \cdots & B(NCORR,NCORR) \end{bmatrix}$$

SVS(I) The variance of the strength of the I^{th} component not included on card 6, above.

SVL(I) The variance of the load, on the I^{th} component not included on card 5, above.

MVS(I) The mean strength of the I^{th} component.

MUL(I) The mean load on the I^{th} component.

TABLE 4. COMPONENT FACTOR OF SAFETY FOR NORMAL-NORMAL AND LOGNORMAL-LOGNORMAL CASES

	Column 1				FORMAT
Card 1	FSR1				(A4)
Card 2a	TITLE "Print case title here"				(A5) (8A10)
Card 3a	TYPE	D3			(A5, A10)
Card 4a	LP	D4			(A5, A10)
Card 5a	PF	D5.1	D5.2	D5.3	(A5, I10, 2F10.0)
Card 6a	PL	D6.1	D6.2	D6.3	(A5, I10, 2F10.0)
Card 7a	PA	D7.1	D7.2	D7.3	(A5, I10, 2F10.0)
Card 8a	VL	D8.1	D8.2	D8.3	(A5, I10, 2F10.0)
Card 9a	VS	D9.1	D9.2	D9.3	(AF, I10, 2F10.0)
Card 10a	XN	D10.1	D10.2	D10.3	(AF, I10, 2F10.0)
Card 11a	RUN				(A3)
Card 12	END				(A3)

The card titles and parameters are described as follows.

TITLE Optional card. Causes the title on the following card to be read and printed as the case title.

TYPE Distribution to be used in the factor of safety computation.

D3 Set to NOR or LOG - Causes program to compute the factor of safety for the normal load - normal strength case or the lognormal load - lognormal strength case.

LP Lines per page.

D4 The integer number of calculations to be printed on a page.

PF Component probability of failure.

D5.1 The number of incremental values of PF where the factor of safety is computed.

D5.2 The lowest value of PF where the factor of safety is computed.

TABLE 4. (Continued)

D5.3 The largest value of PF where the factor of safety is computed.

(If D5.1 = 0, then the factor of safety is computed for PF = D5.2, only. Otherwise, the factor of safety is computed at D5.1 + 1 equally spaced values between D5.2 and D5.3, inclusive.)

PL — The probability that the random input load will not exceed the design limit load.

D6.1
D6.2 Same description as D5.1 through D5.3 for PL rather
D6.3 than PF.

PA — The probability that the random component strength will exceed the design strength.

D7.1
D7.2 Same description as D5.1 through D5.3 for PA rather
D7.3 than PF.

VL — The coefficient of variation of the component load

D8.1
D8.2 Same description as D5.1 through D5.3 for VL rather
D8.3 than PF.

VS — The coefficient of variation of the component strength.

D9.1
D9.2 Same description as D5.1 through D5.3 for VS rather
D9.3 than PF.

XN — This is a coefficient of uncertainty that is used as a multiplier of the structural load. If the accuracy of the estimates of the load parameters is in doubt, set this equal to a number greater than 1.0. This causes the reliability to be conservative. When we are reasonably sure that there is no error in the load parameters, we set XN equal to 1.0.

D10.1
D10.2 Same description as D5.1 through D5.3 for XN rather
D10.3 than PF.

Multiple cases can be run by repeating any or all of the cards 2a through 10a, and following these by a RUN card. These repeated cards must be placed between cards 11a and 12. Only those parmeters which are to be changed need to be listed.

Numerical examples demonstrating the use of this option of FSR are given in Reference 3.

Option 2 of the FSR program is similar to the option described above, except that probability distributions for load and strength other than normal-normal and lognormal-lognormal are available. Reference 3 gives a complete list of the probability distributions available for use in factor of safety computations. Three of these are the normal, lognormal, and extreme value Type I distributions. In this option, similar information to that listed above is provided, except that here, the probability of failure cannot be specified. Rather, it is computed by the program and given as output in addition to the factor of safety. The cards necessary to run Option 2 are listed in Table 5. An explanation for each variable is given immediately following the table. The input format is listed with each card.

Multiple cases can be run by repeating any or all of the cards 2a through 12a, and following these by a RUN card. These repeated cards must be placed between cards 14a and 15. Only those parameters which are to be changed need be listed.

Numerical examples demonstrating the use of this option are given in Section IV.

TABLE 5. COMPONENT FACTOR OF SAFETY FOR MIXED LOAD-STRENGTH CASES

	Column 1				Format
Card 1	FSR2				(A4)
Card 2a	TITLE "Print case title here"				(A5) (8A10)
Card 3a	NPS	D3			(A5, I10)
Card 4a	NPL	D4			(A5, I10)
Card 5a	NPI	D5			(A5, I10)
Card 6a	PDS	D6.1	D6.2	D6.3	(A5, A10, 2F10.0)
Card 7a	PDL	D7.1	D7.2	D7.3	(A5, A10, 2F10.0)
Card 8a	PA	D8			(A5, F10.0)
Card 9a	PL	D9			(A5, F10.0)
Card 10a	XN	D10			(A5, F10.0)
Card 11a	NI	D11			(A5, I10)
Card 12a	IU	D12			(A5, F10.0)
Card 13a	IL	D13			(A5, F10.0)
Card 14a	RUN				(A3)
Card 15	END				(A3)

Card titles and parameters are described as follows.

TITLE — Optional card. Causes the title on the following card to be read and printed as the case title.

NPS — Optional card. Sets the option for printing a table of the strength distribution.

D3 — The ordinal number indicating which points in the table of computed values are printed; i.e., every D3th value is printed.

NPL — Optional card. Sets the option for printing a table of the load distribution.

D4 — The same as D3.

NPI — Optional card. Sets the option for printing a table of the integration calculations.

D5 — The same as D3.

TABLE 5. (Continued.)

PDS		Strength probability distribution.
	D6.1	Selects distribution. Must be set to NORMAL LOGNORMAL EXTREMAL1
		(For other distributions see Reference 3.)
	D6.2	Mean of the strength random variable.
	D6.3	Coefficient of variation of the strength random variable.
PDL		Load probability distribution.
	D7.1	Same as D6.1
	D7.2	Mean of the load random variable.
	D7.3	Coefficient of variation of the load random variable.
PA		The probability that the random component strength will exceed the design strength.
	D8	Probability value.
PL		The probability that the random input load will not exceed the design limit load.
	D9	Probability value.
XN		Coefficient of uncertainty, described following Table 4.
	D10	Value of coefficient of uncertainty.
NI		Integration constant.
	D11	The number of intervals to be used in the numerical integration.
IU		Integration constant.
	D12	Upper limit of the numerical integration.
IL		Integration constant.
	D13	Lower limit of the numerical integration.

IV. NUMERICAL EXAMPLES

Three numerical examples are presented in this section. The first example solves a reliability problem which was first solved by the author in 1977 using a totally different reliability analysis approach. It was solved later by Shevlin and Rodeman using FSR and a certain set of assumptions. The second problem considers the reliability of one part in a wing pylon supporting six Air Launched Cruise Missiles. The third problem shows the effect on factor of safety caused by use of various assumptions for the load and strength probability distributions.

1. RELIABILITY OF THE MAU-12 BOMB RACK

The reliability of the MAU-12 bomb rack is analyzed in this example. The approach used in analysis in this problem is a simplified form of that used in Reference 8; therefore, the problem description follows that of Reference 8. Figure 10 shows an overall view of the rack, and details are shown in Figures 11 through 15. The rack is loaded at 14 different points; the load points and moments of the random load are listed in Table 6. Loads 15 through 18 are linear combinations of the other loads. The load at each point is a stationary random process and is the result of aerodynamic turbulence excitation of the bomb. The random process loads are discussed in detail in Reference 8. Using the procedures of paragraph II-2, find the moments of the peak loads. These are listed in Table 6 and are obtained from information presented in References 8 and 9. Specifically, the values of σ_x, μ_x and $E[N_+(0)]$ are taken from References 8 and 9. For each load, u is found using Equations 79 and 80. The spectral densities are found in Reference 8; the input duration, T, is 10 s; and α is found using Equation 81. These are used to find μ_z and σ_z in Equation 78;

8. Meyer, S. D. and Paez, T. L., "Measurement of Suspension Loads and Determination of Suspension Reliability for a Store in the F-111 Weapons Bay," Fourth Aircraft/Stores Compatibility Symposium Proceedings, Volume 2, JTCG/MD WP #12, Sponsored by: Joint Technical Coordinating Group for Munitions Development, Fort Walton Beach, Florida, October 1977.

9. Shevlin, B. E. and Rodeman, R., "Evaluation of the Design Factors Program," AFWL-TR-79-89, Air Force Weapons Laboratory, Kirtland AFB, NM, October 1979.

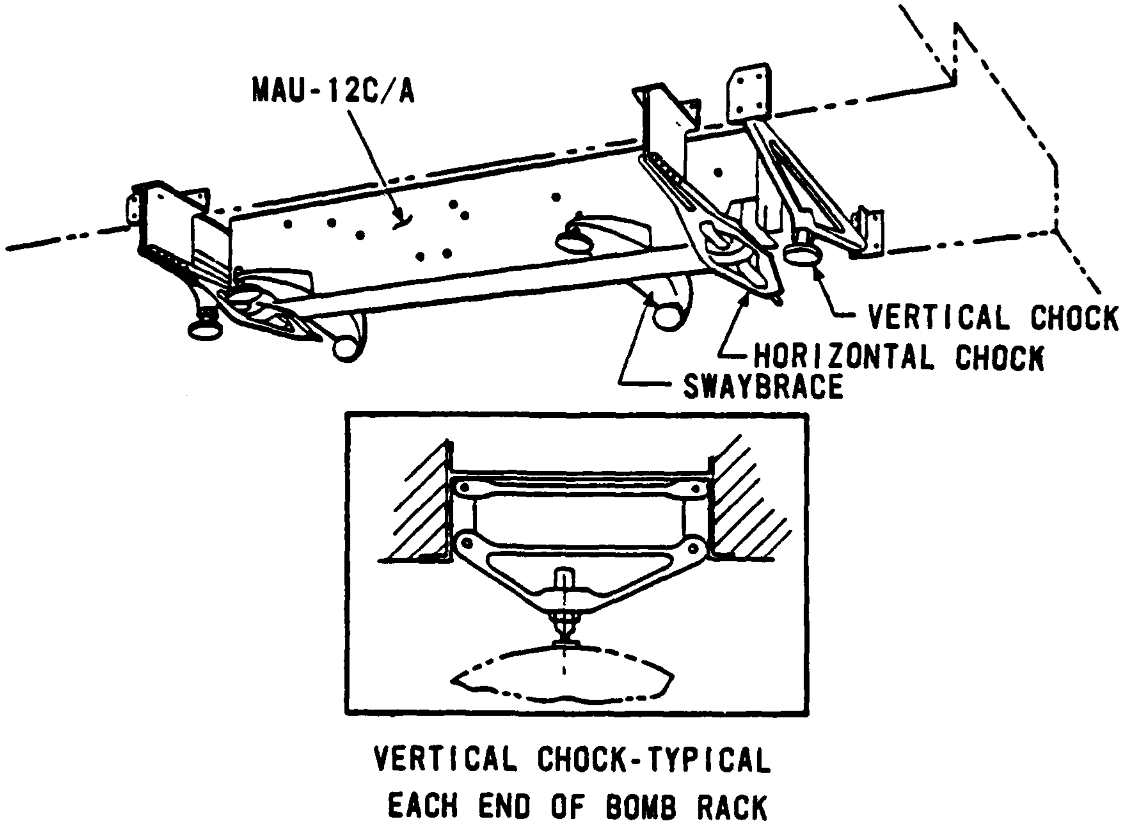

Figure 10. Overall view of the MAU-12 bomb rack.

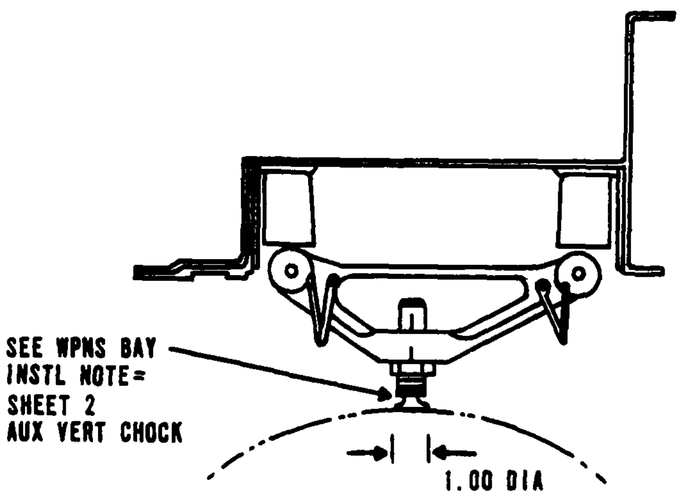

Figure 11. Vertical chock (drawing courtesy of General Dynamics).

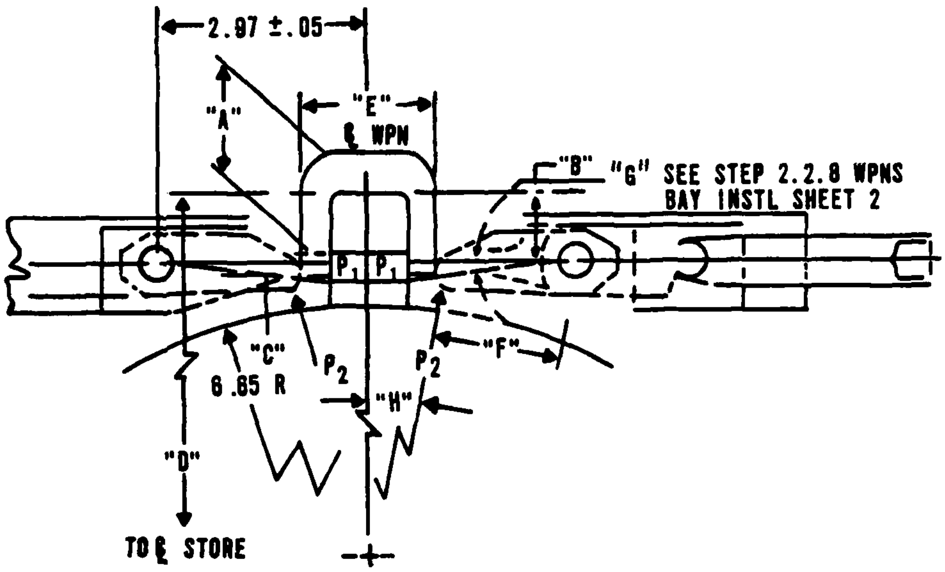

Figure 12. Weapon-loaded suspension lug (General Dynamics).

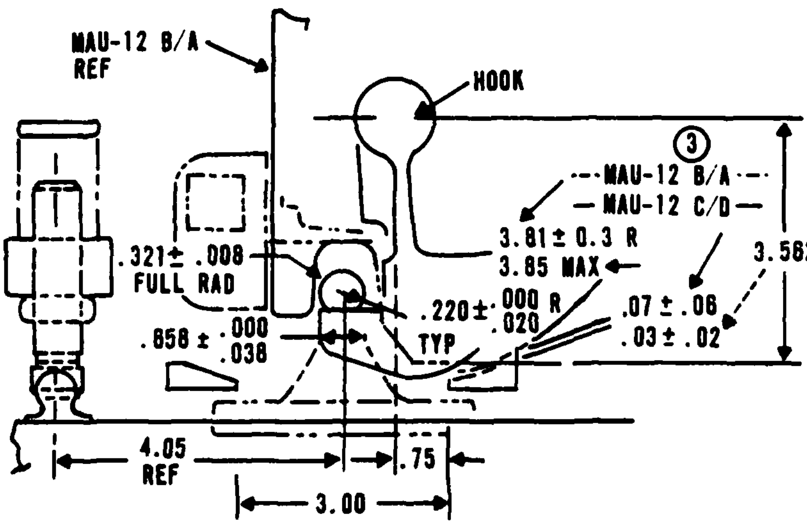

Figure 13. Same lug rotated 90 degrees counterclockwise (General Dynamics).

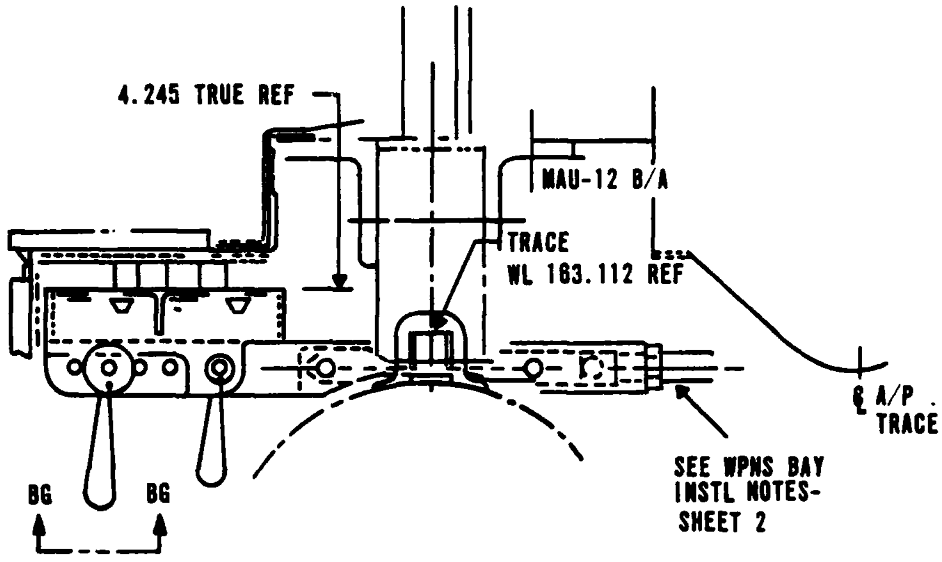

Figure 14. Horizontal chock (General Dynamics).

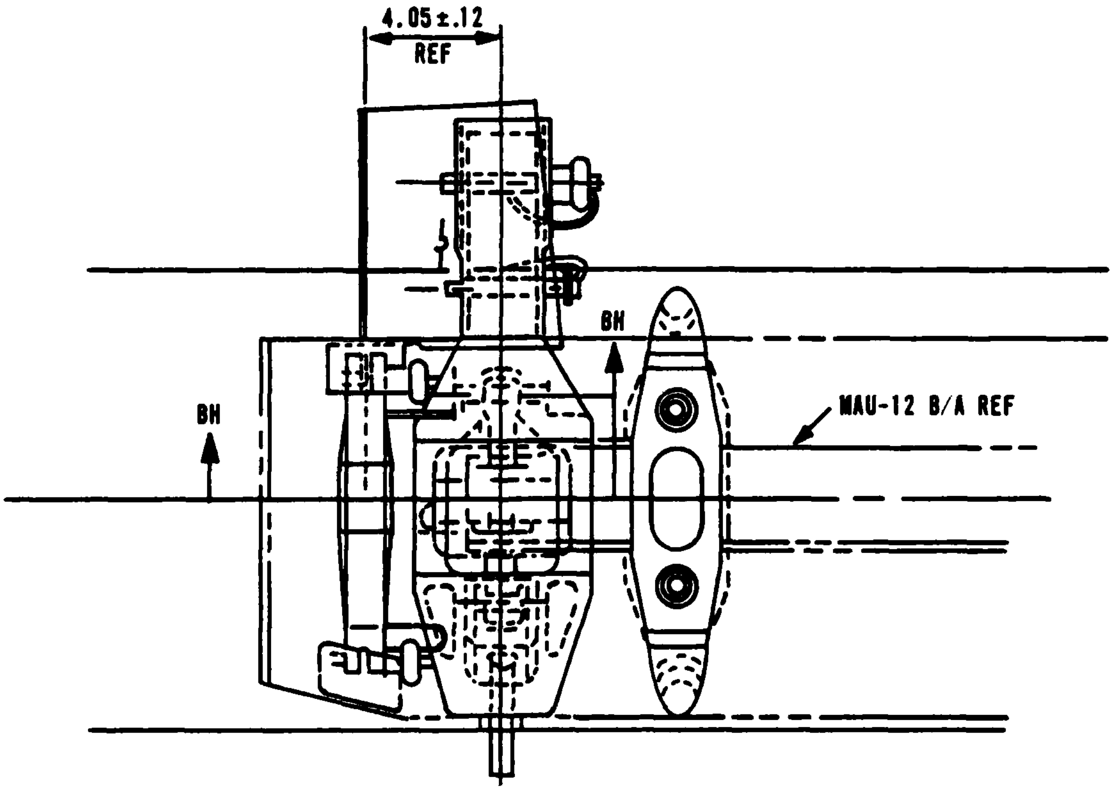

Figure 15. Top view of (from left) vertical chock, horizontal chock, and swaybrace (General Dynamics).

TABLE 6. MAU-12 LOADS MOMENTS

Load No.	Description	σ_X (Kips)	μ_X (Kips)	$E[N_+(0)]$	u	α	μ_p (Kips)	σ_p (Kips)
1	Fwd left pawl	0.210	1.957	45.3	3.5	3.76	2.72	0.072
2	Fwd rt pawl	0.225	3.012	38.4	3.45	3.71	3.82	0.078
3	Fwd vert Chock	0.455	0.093	33.7	3.41	3.67	1.72	0.159
4	Fwd left sway-brace	0.180	3.455	41.1	3.47	3.73	4.11	0.062
5	Fwd rt swaybrace	0.180	3.659	43.0	3.48	3.79	4.31	0.062
6	Fwd lug lateral	0.395	0.488	32.7	3.4	3.66	1.89	0.138
7	Fwd lug vertical	0.565	6.975	32.2	3.4	3.66	8.99	0.198
8	Aft left pawl	0.425	3.401	28.5	3.37	3.63	4.90	0.150
9	Aft rt pawl	0.195	3.237	86.6	3.44	3.70	3.40	0.068
10	Aft vert Chock	0.310	0.196	30.7	3.38	3.64	1.29	0.109
11	Aft left sway-brace	0.205	3.461	38.0	3.45	3.71	4.20	0.071
12	Aft rt swaybrace	0.315	4.377	30.0	3.38	3.64	5.49	0.111
13	Aft lug lateral	0.500	0.425	29.0	3.37	3.63	2.19	0.177
14	Aft lug vertical	0.535	6.943	32.0	3.4	3.66	8.85	0.187
15	Moment at fwd end of rack	1.175 (in Kips)	0	42.0	3.48	3.74	4.27	0.403
16	Moment at aft end of rack	1.725 (in Kips)	0	32.8	3.4	3.66	6.14	0.604
17	Fwd horizontal chock pins	0.155 (in Kips)	0	41.5	3.47	3.73	0.56	0.053
18	Aft horizontal chock pins	0.235 (in Kips)	0	30.7	3.38	3.64	0.83	0.083

σ_x^2 is found using Equation 75. The mean and variance of the nonnormalized peak loads are found using Equation 91.

The potential failure points are determined by considering each structural element separately. The swaybrace shown in Figure 10 is comprised of three fundamental elements; the swaybrace arms, the 1/2-in-diam bolt supporting the swaybrace pad, and the bomb casing. The casing of the MAU-12 rack is also considered a structural element. The vertical chock (Figure 11) is comprised of five fundamental elements: the 5/16 in-diam ball-lok pin, the 3/8 in-diam ball-lok pin, the 1/2 in-diam aircraft bolt, the bomb casing, and the brace supporting the vertical chock. The lug and bomb, shown in Figures 12 and 13 are considered as a single element of the structure. The hook seen in Figure 13 is also a single structural element. The horizontal chock, Figure 14, is comprised of six fundamental structural elements: the 7/16-in-diam bolts which support the pawls, the 1/2-in-diam bolt which tightens down the inner pawl, and the lock frame.

Note that, instead of considering the bomb casing as a single structural element subject to several loads, each section where the bomb casing is loaded is considered to be a structural element for the device which produces the load; i.e., the section of the bomb casing near the vertical chock is considered to be an element in the vertical chock assembly, whereas the section of the casing near the swaybrace areas is considered to be part of the swaybrace assembly.

With the exception of the lugs and the rack frame, each structural component has one load and one failure point. The lugs must carry both vertical and horizontal loads. The rack frame is loaded by the hooks and the swaybrace arms. A list of individual structural components is given in Table 7 and the loads on each component, from Table 6, are also listed.

The strength data were acquired from a variety of sources. These included handbooks, experiments, manufacturers, and finite-element analysis. In particular, the failure loads of the ball-lok pins were obtained from manufacturer's data, whereas the failure loads of the lugs and bomb casing were determined experimentally. The failure loads of the hooks and the various bolts were obtained from handbooks and/or specification. The failure loads of the swaybrace arms, and the horizontal chock frame and the rack frame (both in the areas which support the hooks and to which torque is applied by the swaybraces)

TABLE 7. MAU-12 ASSEMBLY COMPONENT LOADING

Component	Description	Loading on Component (See Table 6)
1	Bolt which tightens down the inner pawl of the horizontal chock	2
2	Frame of the forward horizontal chock	2
3	Bolt supporting the inner pawl of the forward horizontal chock	2
4	Bomb casing at forward vertical chock pad	3
5	5/16-in ball-lok pin in the forward vertical chock	3
6	3/8-in ball-lok pin in the forward vertical chock	3
7	Frame of the forward vertical chock	3
8	Bolt supporting the forward vertical pad	3
9	Bomb casing at the fwd left swaybrace pad	4
10	Fwd left swaybrace bolt	4
11	Fwd left swaybrace arm	4
12	Bomb casing at the fwd right swaybrace	5
13	Fwd right swaybrace bolt	5
14	Fwd right swaybrace arm	5
15	Fwd lug (vertical load)	7
16	Fwd hook (vertical load)	7
17	Rack frame supporting fwd hook	7
18	Bolt which tightens down the inner pawl of the aft horizontal chock	9
19	Frame of the aft horizontal chock	9
20	Bolt supporting the inner pawl of the aft horizontal chock	9
21	Bomb casing at the aft vertical chock pad	10
22	5/16-in ball-lok pin in the aft vertical chock pad	10
23	3/8-in ball-lok pin in the aft vertical chock pad	10

TABLE 7. (Continued.)

Component	Description	Loading on Component (See Table 6)
24	Frame of the aft vertical chock	10
25	Bolt supporting the aft vertical chock pad	10
26	Bomb casing at the aft left swaybrace pad	11
27	Aft left swaybrace bolt	11
28	Aft left swaybrace arm	11
29	Bomb casing at the aft right swaybrace pad	12
30	Aft right swaybrace bolt	12
31	Aft right swaybrace arm	12
32	Aft lug (vertical load)	14
33	Aft hook (vertical load)	14
34	Rack frame supporting the aft hook	14
35	7/16-in ball-lok pin in the forward horizontal chock	17
36	1/2-in ball-lok pin in the forward horizontal chock	17
37	7/16-in ball-lok pin in the aft horizontal chock	18
38	1/2-in ball-lok pin in the aft horizontal chock	18
39	Bolt supporting the outer pawl of the fwd horizontal chock	1
40	Forward lug (lateral load)	6
41	Bolt supporting the outer pawl of the aft horizontal chock	8
42	Aft lug (lateral load)	13
43	Rack casing subject to twisting at the fwd swaybrace	15
44	Rack casing subject to twisting at the aft swaybrace	16

were determined from the handbook material strength values and elementary solid mechanics formulas. However, the vertical chock frame was not suitable for an elementary stress analysis and was analyzed by the ADINA finite-element code. The finite-element analysis used to define the strength of the vertical chock frame is detailed in Reference 7. The mean and the variance of the strength of each component are listd in Table 8.

The moments of the peak load on each structural component are also listed in Table 8, based on the information given in Tables 6 and 7.

To determine which components should be included in the reliability analysis, the reliability index for each component is computed using Equation 117. The lowest reliability index is 8.0, and this applies to the components numbered 16 and 33, the fore and aft hooks. Since the value, 8.0, does not appear in Table 2, the formula in Table 2 may be used to determine the range of reliability indices which must be considered in the analysis. It is found that q is 8.58; therefore, components with reliability indices in the interval (8.0, 8.58) must be included in the analysis. This means that only the two elements, 16 and 33, need be included in the analysis. To be conservative, include those elements in the range (8.0, 10.0) in the FSR reliability analysis.

Even though some of the input loads and component strengths are correlated, in this problem it is assumed that all loads and strengths are uncorrelated. (Loads are correlated here since some loads pass through one component to load another. Strengths are correlated since some single components, loaded in different directions or different modes, are assumed to be two separate components.) The inputs used in the FSR program are listed in Table 9. Loads and strengths are assumed to be normally distributed. The computer program output indicated that the structural reliability is 1.0000000000000000.

It is easy to solve this problem by hand if we consider only the two least reliable elements, 16 and 33, the forward and aft hooks. Using the data in Table 8 and Equation 51, the reliability of one hook is

$$R = \Phi(8.0) = 1 - (0.5 \times 10^{-18}) \tag{131}$$

The joint reliability of two components like this is simply the square of this quantity. The predicted structural reliability is

$$R = [1 - (0.5 \times 10^{-18})]^2 \cong 1 - 10^{-18} \tag{132}$$

TABLE 8. COMPONENT LOAD AND STRENGTH MOMENTS

Component Number	Mean Strength (kips)	Variance of Strength ($kips^2$)	Mean Load (kips)	Variance of Load ($kips^2$)	q
1	83.8	44.9	3.82	0.0052	11.9
2	90.0	20.3	3.82	0.0052	17.8
3	20.0	1.0	3.82	0.0052	16.1
4	10.0	0.3	1.72	0.0253	14.5
5	13.8	0.5	1.72	0.0253	16.7
6	21.3	1.1	1.72	0.0253	18.5
7	19.5	3.8	1.72	0.0253	9.1
8	83.8	44.9	1.72	0.0253	12.2
9	25.0	1.6	4.11	0.0038	16.5
10	83.8	44.9	4.11	0.0038	11.9
11	64.6	41.7	4.11	0.0038	9.4
12	25.0	1.6	4.31	0.0038	16.3
13	83.8	44.9	4.31	0.0038	11.9
14	64.6	41.7	4.31	0.0038	9.3
15	25.0	1.6	8.99	0.0392	12.5
16	45.0	20.3	8.99	0.0392	8.0
17	28.0	2.0	8.99	0.0392	13.3
18	83.8	44.9	3.94	0.0046	12.0
19	90.0	20.3	3.94	0.0046	19.1
20	20.0	1.0	3.94	0.0046	16.0
21	10.0	0.3	1.29	0.0119	15.6
22	13.8	0.5	1.29	0.0119	17.5
23	21.3	1.1	1.29	0.0119	19.0
24	19.5	3.8	1.29	0.0119	9.3
25	83.8	44.9	1.29	0.0119	12.3
26	25.0	1.6	4.20	0.0050	17.2
27	83.8	44.9	4.20	0.0050	11.9
28	64.6	41.7	4.20	0.0050	9.3
29	25.0	1.6	5.49	0.0123	15.4
30	83.8	44.9	5.49	0.0123	11.7

TABLE 8. (Continued)

Component Number	Mean Strength (kips)	Variance of Strength ($kips^2$)	Mean Load (kips)	Variance of Load ($kips^2$)	q
31	64.6	41.7	5.49	0.0123	9.1
32	25.0	1.6	8.85	0.0350	12.6
33	45.0	20.3	8.85	0.0350	8.0
34	28.0	2.0	8.85	0.0350	13.4
35	28.6	2.0	0.56	0.0028	19.8
36	37.4	3.5	0.56	0.0028	19.7
37	28.6	2.0	0.83	0.0069	19.6
38	37.4	3.5	0.83	0.0069	19.5
39	20.0	1.0	2.72	0.0052	17.1
40	10.0	0.3	1.89	0.0190	14.3
41	20.0	1.0	4.90	0.0225	14.9
42	10.0	0.3	2.19	0.0313	13.6
43	200.0 (in-k)	400.0 $(in\text{-}k)^2$	4.27 (in-k)	0.1624 $(in\text{-}k)^2$	9.8
44	200.0 (in-k)	400.0 $(in\text{-}k)^2$	6.14 (in-k)	0.3648 $(in\text{-}k)^2$	9.7

TABLE 9. INPUT TO THE FSR COMPUTER PROGRAM FOR THE MAU-12 BOMB RACK RELIABILITY PROBLEM

Card 1	FSR 3									
Card 2	1	10	0	1	0					
Card 3	0									
Card 4	0									
Card 5	(skip)									
Card 6	(skip)									
Card 7	3.8	41.7	41.7	20.3	3.8	41.7	41.7	20.3	400.0	400.0
Card 8	0.0253	0.0038	0.0038	0.0392	0.0119	0.0050	0.0123	0.0350	0.1624	0.3648
Card 9	19.5	64.6	64.6	45.0	19.5	64.6	64.6	45.0	200.0	200.0
Card 10	1.72	4.11	4.31	8.99	1.29	4.20	5.49	8.85	4.27	6.14

This result agrees with the result obtained using FSR. Both results show that there is practically no possibility that the MAU-12 bomb rack will yield at any point due to the aerodynamic input.

If it had been assumed that failure of the structure occurs at some point lower than the yield level, then the predicted structural reliability would be lower. A practical reason why this assumption might be made is the following. When a structural component is excited over a long period of time, it may fail in fatigue. The load stress level which causes failure in fatigue is a stress which is greater than the endurance limit and lower than the yield level. If the analyst wishes to obtain extremely conservative results, he can assume that failure occurs when the load stress surpasses the endurance limit. Less conservative results can be obtained by assuming that failure occurs at a level between the endurance limit and the yield level.

This numerical example demonstrates an important point in reliability analysis. That is, we are often interested in finding the reliability of a structural system which is composed of many structural components, and which is, itself, a subsystem of a large system. The MAU-12 bomb rack is a subsystem of the airplane to which it is attached, and it is composed of many structural components. When it can be assumed that the subsystem and the larger system of which it is a part, act independently, then the overall system reliability is the product of the subsystem reliability and the reliability of the remainder of the system. Even when this assumption is not strictly correct, the product of reliabilities provides a lower bound on the true reliability of the overall system.

The following example shows how the reliability of a component in the B-52 ALCM pylon can be calculated.

2. RELIABILITY OF ALCM PYLON COMPONENT

A pylon for use on B-52G aircraft for carrying ALCMs (Air Launched Cruise Missiles) has been designed. A deterministic analysis of the loads to be carried by the pylon is presented in Reference 10. A summary of deterministic

10. Heffron, C. J. and Clements, R., "Preliminary Loads Report, CM1," CDRL Seq. No. 066 DI-S-30588, Boeing Wichita Company, FSCM No. 82918, August 1978.

structural analysis on the pylon is presented in Reference 11. This analysis calculates the reliability of the forward pylon/support fitting lug pin. The reason we analyze this element is that deterministic analysis has shown that the lug pin has a margin of safety equal to 0.02. The lug pin is made from 15-5 PH stainless steel. The properties of this material are given in Reference 12. The material strength properties are not presented on a statistical basis; rather they are presented as "producers guaranteed minimum tensile properties." The material properties listed in Reference 12 for this material are given in Table 10. One can infer statistical characteristics for the strength of this material as follows. Assume, first, that the strength properties are values above which 99 percent of the test outcomes are expected to fail. Second, we assume that the coefficients of variation of the strength properties are similar to those of other materials for which statistical data are available. From Reference 10 the coefficients of variation of ultimate tensile strength, F_{tu}, tensile yield strength, F_{ty}, and shear strength, F_{su}, for stainless steels which are statistically characterized are given by the average values:

$$\text{For } F_{tu} \; : \; \text{cov} = 0.034 \tag{133a}$$

$$\text{For } F_{ty} \; : \; \text{cov} = 0.067 \tag{133b}$$

$$\text{For } F_{su} \; : \; \text{cov} = 0.038 \tag{133c}$$

With these assumptions and Equation 133, the statistical strength properties of 15-5 PH stainless steel can be found. These are listed in Table 10. The mean strength values have been multiplied by a factor of 0.95 to account for a 160°F environment. A sketch of the pin and the lugs which load it is shown in Figure 16. An analysis contained in Reference 9 shows that the values of a and b, in the sketch, which maximize the moment in the pin are

$$a = 0.12 \text{ in} \qquad b = 0.0 \text{ in} \tag{134a}$$

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11. Roberts, H. M., "Pylon Stress Analysis, Vol. I," CDRL Seq. No. 069 (DI-S-3581), Boeing Wichita Company, FSCM No. 82918, August 1978.

12. MIL-HDBK-5C, Military Standardization Handbook, Metallic Materials and Elements for Aerospace Vehicle Structures, Vols. 1 and 2, Department of Defense, Washington, D.C., 20025.

TABLE 10. MATERIAL PROPERTIES FOR 15-5 PH STAINLESS STEEL

SPECIFICATION	AMS5659
FORM	BARS AND FORGINGS
CONDITION	H1025
THICKNESS OR DIAMETER	LESS THAN 12 IN
BASIS	S
TENSILE ULTIMATE	F_{tu} = 155 ksi
TENSILE YIELD	F_{ty} = 145 ksi
COMPRESSIVE YIELD	F_{cy} = 143 ksi
SHEAR ULTIMATE	F_{su} = 97 ksi
BEARING ULTIMATE	F_{bru} = 220 ksi (e/D = 1.5) 285 ksi (e/D = 2.0)
BEARING YIELD	F_{bry} = 189 ksi (e/D = 1.5) 222 ksi (e/D = 2.0)
FAILURE STRAIN	e = 12 PERCENT (L DIRECTION) 8 PERCENT (T DIRECTION)
ELASTIC MODULUS	E = 28.5×10^3 ksi
ELASTIC MODULUS (COMPR)	E_c = 29.2×10^3 ksi
SHEAR MODULUS	G = 11.2×10^3 ksi
POISSONS RATIO	V = 0.27

Statistical Properties of 15-5 PH Stainless Steel (160°F)

	Average	Coefficient of Variation
Tensile Ultimate Stress	154.7	0.034
Tensile Yield Stress	151.8	0.067
Ultimate Shear Stress	101.1	0.038

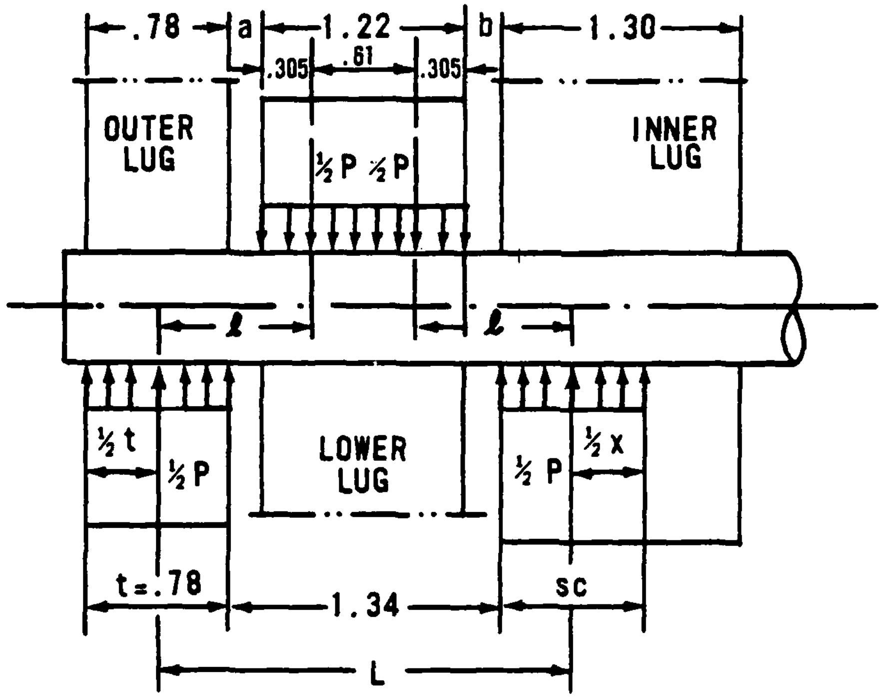

Figure 16. Forward pylon/support fitting lug pin for B-52 ALCM pylon.

These imply that ℓ and x, in the sketch, have values

$$\ell = 0.815 \text{ in} \qquad x = 0.102 \text{ in} \tag{134b}$$

When it is assumed that the lugs supporting the ends of the pin act as simple supports, the maximum moment and shear in the lug pin are

$$M = 0.5\ P\ell \tag{135a}$$

$$S = 0.5\ P \tag{135b}$$

where M is moment, S is shear, and P is the load on the pin. It has been determined in References 10 and 11 that the most severe load on this particular element occurs due to a lateral 2 g acceleration. The total load exciting a bending response in the pin is

$$P = 151.6 \text{ kips} \tag{136}$$

Accordingly, the maximum applied bending moment and shear load are

$$M = 61.8 \text{ in - k} \tag{137a}$$

$$S = 75.9 \text{ kips} \tag{137b}$$

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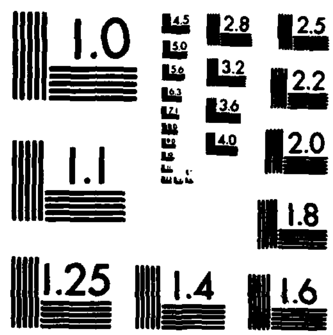

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These are the values obtained in the analysis of Reference 11. It will be assumed that the loads are deterministic.

Following the methodology used in Reference 11 for obtaining the ultimate moment resistance of the lug pin in bending, write

$$M_R = F_B Z \tag{138}$$

Where M_R is moment resistance, F_B is the bending modulus of rupture, and Z is the section modulus given by

$$Z = \frac{1}{32} \pi d^3 = 0.329 \text{ in}^3 \tag{139}$$

where d = 1.4965 is the pin diameter. F_B is given by

$$F_B = F_{tu} + F_{ty} (K - 1) \tag{140}$$

where K = 1.7 is a constant which depends on cross-sectional shape. F_B can be simplified to

$$F_B = F_{tu} + 0.7 F_{ty} \tag{141}$$

F_B is a random variable which is the sum of two other random variables, F_{tu} and 0.7 F_{ty}. The mean and the variance of F_B are

$$\mu_{F_B} = \mu_{F_{tu}} + 0.7 \mu_{F_{ty}} = 261.0 \tag{142a}$$

$$\sigma_{F_B}^2 = \sigma_{F_{tu}}^2 + 0.49 \sigma_{F_{ty}}^2 = 86.86 \tag{142b}$$

where F_{tu} and F_{ty} are assumed to be independent. Assume that the bending modulus of rupture, F_B, is normally distributed; therefore, the random variable is completely specified. (Mean, variance, distribution). The statistical moments (mean and variance) of the structural moment resistance of the lug pin in bending, M_R, can be obtained using Equations 138 and 142. These are

$$\mu_{M_R} = 0.329 \mu_{F_B} = 85.9 \tag{143a}$$

$$\sigma_{M_R}^2 = 0.108 \sigma_{F_B}^2 = 9.40 \tag{143b}$$

Since F_B is a normal random variable, M_R is also normal.

The shear resistance of the lug pin is

$$S_R = F_{su}\, A \tag{144}$$

where A = 1.759 in^2 is the cross-sectional area of the pin. S_R is a random variable and is a simple linear function of F_{su}. The mean and variance of S_R are

$$\mu_{S_R} = 1.759\, \mu_{F_{su}} = 177.8 \text{ kips} \tag{145a}$$

$$\sigma_{S_R}^2 = 3.094\, \sigma_{F_{su}}^2 = 45.7 \text{ (kips)}^2 \tag{145b}$$

We assume that the shear resistance is a normal random variable.

In the reliability analysis of the lug pin, assume that the pin acts as two components, one in bending and one in shear. The shear load and the bending moment are perfectly correlated, since they both depend on P. It would be reasonable to assume that the shear and bending resistance are strongly correlated; but, in the preliminary analysis, we assume that all these items are uncorrelated. Since only two components are involved, they are both considered in the analysis. As stated earlier, the moment and shear loads would be considered deterministic. This can be interpreted as meaning that the applied moment and shear are random variables with means

$$\mu_M = 61.8 \text{ in - k} \tag{146a}$$

$$\mu_S = 75.9 \text{ kips} \tag{146b}$$

and zero variances

$$\sigma_M^2 = 0 \tag{147a}$$

$$\sigma_S^2 = 0 \tag{147b}$$

A deterministic constant is a random variable, with mean equal to the constant, and zero variance. We assume that the loads and strength are normal random variables. Then the structural reliability is computed using Equations 57 and 121.

$$R = R_1 \cdot R_2 = \Phi\left(\frac{85.9 - 61.8}{\sqrt{9.40}}\right) \times \Phi\left(\frac{177.8 - 75.8}{\sqrt{45.7}}\right) \tag{148}$$

$$= (1 - 2 \times 10^{-18})\ (1 - 2.3 \times 10^{-64})$$

$$= (1 - 2 \times 10^{-18})$$

The reliability of the system is very high. For the situation considered, no practical possibility of failure exists, even though the margin of safety is very low.

A somewhat more accurate result could be obtained by taking into account the correlation between the moment and shear loads. The result of that analysis would yield a higher reliability than that given above.

3. COMPARISON OF LOAD AND STRENGTH PROBABILITY DISTRIBUTIONS

It was pointed out at the end of paragraph II-1 that, when different probability distributions are assumed to govern the load and strength of a structural component, the computed point reliabilities for the component may differ significantly. To demonstrate this point further, consider the system component which has random strength with mean value 25.0 kips and coefficient of variation 0.1, and load with mean value 10.0 kips and coefficient of variation 0.2. The allowable load for the component is defined so that the probability that the actual load does not exceed the allowable is 0.99. The design strength is defined in such a way that the probability that the actual strength exceeds the design strength is 0.99. Given this information, the reliability and factor of safety between design strength and allowable load are computed for every combination of the assumed load and strength distributions which can be obtained using the normal, lognormal, and extreme value Type I distributions. Table 11 defines those combinations of distributions that can be used and shows the factor of safety and reliability results that were obtained using the computer program FSR2. The input used to run FSR2 follows the specification given in paragraph III-2. The input for the specific problem listed above is given in Table 12.

TABLE 11. FACTORS OF SAFETY AND RELIABILITIES OBTAINED USING VARIOUS LOAD AND STRENGTH ASSUMPTIONS

Case No.	Load cdf	Strength cdf	Factor of Safety	Reliability
1	Normal	Normal	1.31	$1-2.20 \times 10^{-6}$
2	Lognormal	Lognormal	1.27	$1-3.43 \times 10^{-5}$
3	Extremal 1	Extremal 1	1.28	$1-1.04 \times 10^{-3}$
4	Normal	Lognormal	1.23	$1-2.27 \times 10^{-5}$
5	Normal	Extremal 1	1.18	$1-1.35 \times 10^{-4}$
6	Lognormal	Normal	1.35	$1-2.13 \times 10^{-5}$
7	Lognormal	Extremal 1	1.21	$1-1.34 \times 10^{-4}$
8	Extremal 1	Normal	1.43	$1-9.51 \times 10^{-4}$
9	Extremal 1	Lognormal	1.34	$1-9.57 \times 10^{-4}$

Load	Mean = 10.0	cov = 0.2
Strength	Mean = 25.0	cov = 0.1

TABLE 12. INPUT FOR EXAMPLE OF PARAGRAPH IV-3

FSR2			
TITLE			
CASE 1	NORMAL	NORMAL	
NPS	10		
NPL	10		
NPI	10		
PDS	NORMAL	25.0	0.1
PDL	NORMAL	10.0	0.2
PA	0.99		
PL	0.99		
XN	1.0		
NI	300.0		
IU	37.5		
IL	0.0		
RUN			
TITLE			
CASE 2	LOGNORMAL	LOGNORMAL	
PDS	LOGNORMAL	25.0	0.1
PDL	LOGNORMAL	10.0	0.2
RUN			
TITLE			
CASE 3	EXTREMAL1	EXTREMAL1	
PDS	EXTREMAL1	25.0	0.1
PDL	EXTREMAL1	10.0	0.2
RUN			
TITLE			
CASE 4	NORMAL	LOGNORMAL	
PDS	NORMAL	25.0	0.1
PDL	LOGNORMAL	10.0	0.2
RUN			
TITLE			

TABLE 12. (Continued.)

```
CASE 5      NORMAL       EXTREMAL1
PDS         NORMAL        25.0          0.1
PDL         EXTREMAL1     10.0          0.2
RUN
TITLE
CASE 6      LOGNORMAL    NORMAL
PDS         LOGNORMAL     25.0          0.1
PDL         NORMAL        10.0          0.2
RUN
TITLE
CASE 7      LOGNORMAL    EXTREMAL1
PDS         LOGNORMAL     25.0          0.1
PDL         EXTREMAL1     10.0          0.2
RUN
TITLE
CASE 8      EXTREMAL1    NORMAL
PDS         EXTREMAL1     25.0          0.1
PDL         NORMAL        10.0          0.2
RUN
TITLE
CASE 9      EXTREMAL1    LOGNORMAL
PDS         EXTREMAL1     25.0          0.1
PDL         LOGNORMAL     10.0          0.2
RUN
END
```

V. DISCUSSION AND CONCLUSION

This report has presented a brief introduction to structural reliability, and a guide for the use of FSR, a structural reliability computer program. In this chapter we discuss sources of statistical data and data requirements and some practical aspects of reliability analysis.

An accurate reliability analysis requires an accurate statistical characterization of the random loads applied to a structure and the random strength of a structure. The strength characteristics of a structure can usually be found through structural analysis when the characteristics of the structural material are known. The statistical moments of the failure and yield stress have been found for many materials. For example, References 12 and 13 contain information on the probabilistic behavior of many materials.

Reference 12 contains information on many materials and presents it in the following way. Let Y be a material strength random variable. (For example, tensile yield stress, or ultimate shear stress, etc.) The handbook presents an A basis strength, y_A, and a B basis strength, y_B. The y_A is chosen so that 99 percent of all material samples tested will have a strength greater than y_A; y_B is chosen so that 90 percent of all material samples tested will have a strength greater than y_B. It is assumed that Y is a normally distributed random variable; therefore, we can write

$$\Phi\left(\frac{y_A - \mu_y}{\sigma_y}\right) = 1 - 0.99 = 0.01 \tag{149a}$$

$$\Phi\left(\frac{y_B - \mu_y}{\sigma_y}\right) = 1 - 0.90 = 0.10 \tag{149b}$$

These equations can be inverted and solved for y_A and y_B.

$$y_A = \sigma_y \, \Phi^{-1}(0.01) + \mu_y = -2.32\sigma_y + \mu_y \tag{149c}$$

$$y_B = \sigma_y \, \Phi^{-1}(0.10) + \mu_y = -1.28\sigma_y + \mu_y \tag{149d}$$

where $\Phi^{-1}(.)$ is the inverse function of the standard normal cdf. The values of this function may be obtained, for example, in Table 1 of Reference 1. Given

13. Haugen, E. B., "Statistical Metals Manual," Aerospace and Mechanical Engineering, The University of Arizona, Tucson, Ariz., 1974.

the y_A and y_B values for a particular material strength, the moments of the strength random variable can be obtained by solving Equations 149a and b, simultaneously. The y_A and y_B values are chosen as 95 percent confidence values. This means, roughly, that we are 95 percent confident that the values chosen are conservative.

Generally, load statistics must be obtained through measurements of a specific load source. This was done, for example, in Reference 8. A few general load sources have been conservatively modeled; for example, earthquake sources (Ref. 14). It is certainly conceivable that a specific load source could be characterized by a thorough study. For example, the peak loads applied to weapons during ground handling operations could be characterized.

There are numerous practical aspects which must be considered in the application of reliability analyses. One of the most important is the need for finite element analyses in reliability studies. Even in an elementary reliability analysis of a complex structure, we must find the structural member loads induced by an external force. This can usually be done accurately with a finite element computer program. Without the use of a finite element program this may be an impossible task. This is especially true when the load and response are dynamic.

Another practical aspect of reliability analysis is specification of a failure level. When true failure of a structure occurs with collapse, then it is very difficult to find the structural reliability against true failure. The reason is that the failure occurs only after the response has become nonlinear in a geometric and material sense. Except when a special research study can be performed, a conservative assumption regarding failure should be made. When the structure under consideration will be forced through only a small number of response cycles, then a reasonable failure level to use is the yield level. This will provide very conservative results if the true failure occurs with collapse. When the structure under consideration will be forced through a large number of cycles, then a fatigue failure may occur. In this case the failure level may be chosen as follows. Estimate the number of cycles of the response. Use the S-N curve for the material to find the stress level which is expected to cause material failure after the specified number of cycles. Use this stress as the failure level for the material. The results obtained using this approach are conservative since the true failure stress is always greater than that assumed to be the failure level, up to the time when the final response cycle is executed. An extremely conservative result can be obtained by using the endurance limit as the failure level.

It is very rare that a structure is subjected only to loads from one random source during its design life. Rather, a structure is subjected to many different random loads from many sources. To determine the reliability of a structure for all these loads, the analyst should find the probability that the structure will survive each load separately. Then, the product of these reliabilities forms a lower bound on the overall structural reliability. When some of the input loads are correlated with one another, then the true reliability is higher than this estimated figure.

Most of the numerical examples presented in this report use an assumption of normality for the structural loads and strength. This is almost always considered a satisfactory approach, at least for preliminary analyses. In the practical examples presentd in Section IV, the preliminary estimates of reliability were so high that no need for further analysis was seen. This will be the case in many practical situations. When the estimated reliability is lower, the analyst may wish to analyze the structural reliability using the lognormal load and strength assumptions. When this is the case, the specific approach used by Shevlin and Rodeman in Reference 7 should be followed. Here the structural strength is modeled using a "lognormal plus a constant" probability distribution.

While the approaches outlined in this report can be used to find estimates for the reliabilities of structures, the probability figures obtained from the analyses should not be taken as exact. The reason is that many assumptions have been used in obtaining the estimates. The most important of these assumptions should be investigated, and alternate analytic approaches that circumvent the need for these assumptions should be sought.

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